THE LAGRANGIAN SOLUTIONS

Àngel Jorba
Departament de Matemàtiques i Informàtica, Universitat de Barcelona Gran Via 585, 08007 Barcelona, Spain

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Contents

Summary

This chapter focuses on the dynamics in a neighborhood of the five equilibrium points of the Restricted Three-Body Problem. The first section is devoted to the discussion of the linear behavior near the five points. Then, the motion in the vicinity of the collinear points is considered, discussing the effective computation of the center manifold as a tool to describe the nonlinear dynamics in an extended neighborhood of these points. This technique is then applied to the Earth-Moon case, showing the existence of periodic and quasi-periodic motions, including the well-known Halo orbits.

Next, the dynamics near the triangular points is discussed, showing how normal forms can be used to effectively describe the motion nearby. The Lyapunov stability is also considered, showing how the stability is proved in the planar case, and why it is not proved in the spatial case. This section also discusses how to bound the amount of diffusion that could be present in the spatial case. Finally, in the last section we focus on the effect of perturbations. More concretely, we mention the Elliptic Restricted Three-Body Problem, the Bicircular problem and similar models that contain periodic and quasi-periodic time-dependent perturbations.

1. Introduction

Let us consider two point masses (usually called primaries) that attract each other according to the gravitational Newton’s law. Let us assume that they are moving in circular orbits around their common center of mass, and let us consider the motion of an infinitesimal particle (here, infinitesimal means that its mass is so small that we neglect the effect it has on the motion of the primaries and we only take into account the effect of the primaries on the particle) under the attraction of the two primaries. The study of the motion of the infinitesimal particle is what is known as the Restricted Three-Body Problem, or RTBP for short.

To simplify the equations of motion, let us take units of mass, length and time such that the sum of masses of the primaries, the gravitational constant and the period of the motion of the primaries are 1, 1 and 2π respectively. With these units the distance between the primaries is also equal to 1. We denote as μ the mass of the smaller primary (the mass of the bigger is then $1-\mu$), $\mu \in \left(0, \frac{1}{2}\right]$.

The usual system of reference is defined as follows: the origin is taken at the center of mass of the primaries, the X-axis points to the bigger primary, the Z-axis is perpendicular to the plane of motion, pointing in the same direction as the vector of angular momentum of the primaries with respect to their common center of mass, and the Y-axis is defined such that we obtain an orthogonal, positive-oriented system of reference. With this we have defined a rotating system of reference, that is usually called synodic. In this system, the primary of mass μ is located at the point $(\mu-1, 0, 0)$ and the one of mass $1-\mu$ is located at $(\mu, 0, 0)$, see Figure 1.

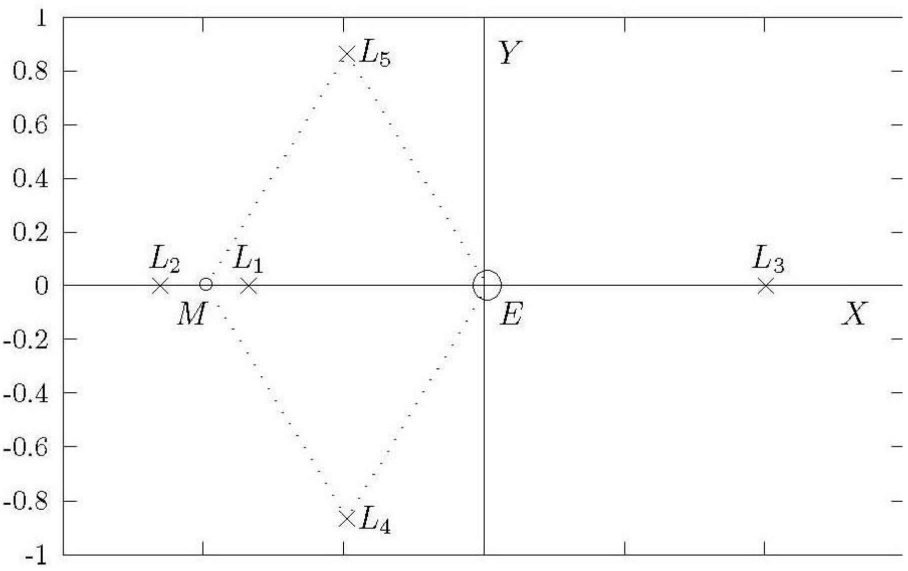

Figure 1. The five equilibrium points of the RTBP. The graphic corresponds to the Earth-Moon case. The unit of distance is the Earth-Moon distance, and the unit of mass is the total mass of the system. In these units, the mass of the Moon is $\mu \approx 0.01215$.

Defining momenta as $P_X = \dot{X} - Y$, $P_Y = \dot{Y} + X$ and $P_Z = \dot{Z}$, the equations of motion can be written in Hamiltonian form. The corresponding Hamiltonian function is

$$H = \frac{1}{2}\left(P_X^2 + P_Y^2 + P_Z^2\right) + YP_X - XP_Y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \qquad (1)$$

being $r_1^2=(X-\mu)^2+Y^2+Z^2$ and $r_2^2=(X-\mu+1)^2+Y^2+Z^2$ (see, for instance, Szebehely (1967) for details).

It is well-known that the system defined by (1) has five equilibrium points. Two of them can be found as the third vertex of the two equilateral triangles that can be formed using the two primaries as vertices (usually called $L_{4,5}$ or Lagrangian points). The other three lie on the X -axis and are usually called $L_{1,2,3}$ or Eulerian points (see Figure 1). A more detailed discussion on the existence of these points can be found in many textbooks, like Szebehely (1967). Note that "our" L_1 and L_2 are swapped with respect to that reference. This lack of agreement for the definition of $L_{1,2}$ is rather common in the literature: usually, books on celestial mechanics use the same notation as in Szebehely (1967) but books on astrodynamics follow the convention we use here.

In this chapter we will focus on the dynamics around these points, especially for examples from the Solar system. We will also comment on the main perturbations that appear in astronomical and astronautical applications and their effects.

2. Linear Behavior

In this section we will first discuss the linearization of the dynamics around the five equilibrium points. The presentation is done in a way that prepares the following sections.

2.1. The Collinear Points

Let us define, for $j=1,2,$ γ_j as the distance from the smaller primary (the one of mass μ) to the point L_j, and γ_3 as the distance from the bigger primary to L_3. It is well-known (see, for instance, Szebehely (1967) that γ_j is the only positive solution of the Euler quintic equation,

$$\gamma_j^5 \mp (3-\mu)\gamma_j^4+(3-2\mu)\gamma_j^3-\mu\gamma_j^2 \pm 2\mu\gamma_j-\mu=0, \qquad j=1,2,$$
$$\gamma_3^5+(2+\mu)\gamma_3^4+(1+2\mu)\gamma_3^3-(1-\mu)\gamma_3^2-2(1-\mu)\gamma_3-(1-\mu)=0,$$

where the upper sign in the first equation is for L_1 and the lower one for L_2. These equations can be solved numerically by the Newton method, using the starting point $(\mu/3)^{1/3}$ for the first equation ($L_{1,2}$ cases), and $1-\frac{7}{12}\mu$ for the second one (L_3 case).

The next step is to translate the origin to the selected point L_j. Moreover, since in Section 3 we will need the power expansion of the Hamiltonian at these points, we therefore perform a suitable scaling in order to avoid fast growing (or decreasing) coefficients. The idea is to have the closest singularity (the body of mass μ for $L_{1,2}$ or the one of mass $1-\mu$ for L_3) at distance 1 (see Richardson, 1980b). As the scalings are not symplectic transformations, let us consider the following process: first we write the differential equations related to (1) and then, on these equations, we perform the following substitution

$$\begin{aligned} X &= \mp\gamma_j x + \mu + \alpha_j, \\ Y &= \mp\gamma_j y, \\ Z &= \gamma_j z, \end{aligned}$$

where the upper sign corresponds to $L_{1,2}$, the lower one to L_3 and $\alpha_1 = -1+\gamma_1$, $\alpha_2 = -1-\gamma_2$ and $\alpha_3 = \gamma_3$. Note that the unit of distance is now the distance from the equilibrium point to the closest primary.

In order to expand the nonlinear terms, we will use that

$$\frac{1}{\sqrt{(x-A)^2+(y-B)^2+(z-C)^2}} = \frac{1}{D}\sum_{n=0}^{\infty}\left(\frac{\rho}{D}\right)^n P_n\left(\frac{Ax+By+Cz}{D\rho}\right),$$

where A, B, C, D, are real numbers with $D^2 = A^2+B^2+C^2$, $\rho^2 = x^2+y^2+z^2$ and P_n is the polynomial of Legendre of degree n. After some calculations, one obtains that the equations of motion can be written as

$$\begin{aligned} \ddot{x} - 2\dot{y} - (1+2c_2)x &= \frac{\partial}{\partial x}\sum_{n>3} c_n(\mu)\rho^n P_n\left(\frac{x}{\rho}\right), \\ \ddot{y} + 2\dot{x} + (c_2-1)y &= \frac{\partial}{\partial y}\sum_{n>3} c_n(\mu)\rho^n P_n\left(\frac{x}{\rho}\right), \qquad (2) \\ \ddot{z} + c_2 z &= \frac{\partial}{\partial z}\sum_{n>3} c_n(\mu)\rho^n P_n\left(\frac{x}{\rho}\right), \end{aligned}$$

where the left-hand side contains the linear terms and the right-hand side contains the nonlinear ones. The coefficients $c_n(\mu)$ are given by

$$c_n(\mu) = \frac{1}{\gamma_j^3}\left((\pm1)^n\mu + (-1)^n\frac{(1-\mu)\gamma_j^{n+1}}{(1\mp\gamma_j)^{n+1}}\right), \qquad \text{for } L_j,\ j=1,2$$

$$c_n(\mu) = \frac{(-1)^n}{\gamma_3^3}\left(1-\mu+\frac{\mu\gamma_3^{n+1}}{(1+\gamma_3)^{n+1}}\right), \qquad \text{for } L_3.$$

In the first equation, the upper signs are for L_1 and the lower one for L_2. Note that these equations can be written in Hamiltonian form, by defining the momenta $p_x = \dot{x} - y$,

$p_y = \dot{y} + x$ and $p_z = \dot{z}$. The corresponding Hamiltonian is then given by

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + yp_x - xp_y - \sum_{n\geq 2} c_n(\mu)\rho^n P_n\left(\frac{x}{\rho}\right). \tag{3}$$

The nonlinear terms of this Hamiltonian can be expanded by means of the well-known recurrence of the Legendre polynomials P_n. For instance, if we define

$$T_n(x,y,z) = \rho^n P_n\left(\frac{x}{\rho}\right), \tag{4}$$

then, it is not difficult to check that T_n is a homogeneous polynomial of degree n that satisfies the recurrence

$$T_n = \frac{2n-1}{n} xT_{n-1} - \frac{n-1}{n}\left(x^2 + y^2 + z^2\right)T_{n-2}, \tag{5}$$

starting with $T_0 = 1$ and $T_1 = x$.

The linearization around the equilibrium point is given by the second order terms of the Hamiltonian (linear terms must vanish) that, after some rearranging, take the form,

$$H_2 = \frac{1}{2}\left(p_x^2 + p_y^2\right) + yp_x - xp_y - c_2x^2 + \frac{c_2}{2}y^2 + \frac{1}{2}p_z^2 + \frac{c_2}{2}z^2. \tag{6}$$

It is not difficult to derive intervals for the values of $c_2 = c_2(\mu)$ when $\mu \in \left[0, \tfrac{1}{2}\right]$ (see Figure 2). As $c_2 > 0$ (for the three collinear points), the vertical direction is an harmonic oscillator with frequency $\omega_2 = \sqrt{c_2}$. Now let us focus on the planar directions, i.e.,

$$H_2 = \frac{1}{2}\left(p_x^2 + p_y^2\right) + yp_x - xp_y - c_2x^2 + \frac{c_2}{2}y^2, \tag{7}$$

where, for simplicity, we keep the name H_2 for the Hamiltonian.

Now, let us define the matrix $\mathbf{M}$ as $\mathbf{J}\,\mathrm{Hess}(H_2)$,

$$\mathrm{M} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 2c_2 & 0 & 0 & 1 \\ 0 & -c_2 & -1 & 0 \end{bmatrix}. \tag{8}$$

The characteristic polynomial is $p(\lambda)=\lambda^4+(2-c_2)\lambda^2+(1+c_2-2c_2^2)$. Calling $\eta=\lambda^2$, we have that the roots of $p(\lambda)=0$ are given by

$$\eta_1=\frac{c_2-2-\sqrt{9c_2^2-8c_2}}{2}, \qquad \eta_2=\frac{c_2-2+\sqrt{9c_2^2-8c_2}}{2}.$$

As $\mu\in\left]0,\frac{1}{2}\right]$ we have that $c_2>1$ that forces $\eta_1<0$ and $\eta_2>0$. This shows that the equilibrium point is a center$\times$center$\times$saddle. Thus, let us define ω_1 as $\sqrt{-\eta_1}$ and λ_1 as $\sqrt{\eta_2}$. For the moment, we do not specify the sign taken for each value (this will be discussed later on).

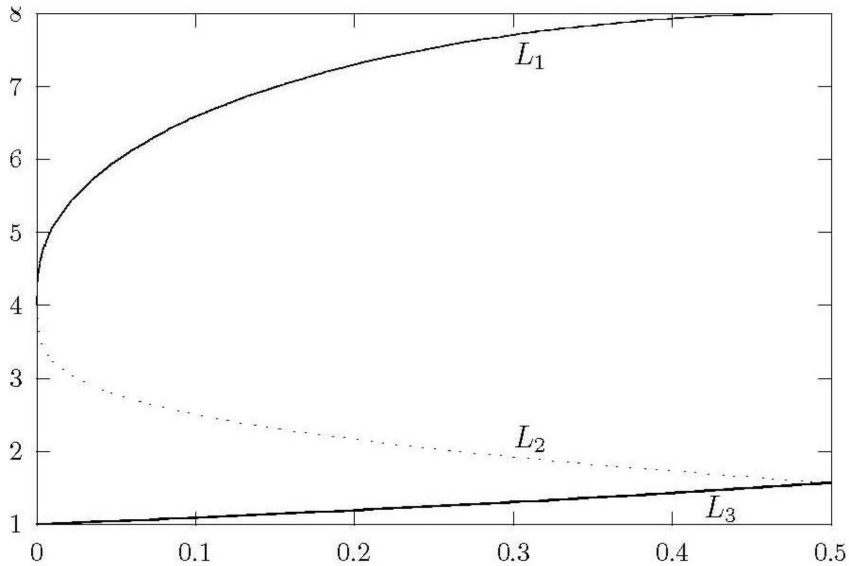

Figure 2. Values of $c_2(\mu)$ (vertical axis), for $\mu\in\left[0,\frac{1}{2}\right]$ (horizontal axis), for the cases $L_{1,2,3}$.

Now, we want to find a symplectic linear change of variables casting (7) into its real normal form (by real we mean with real coefficients) and, hence, we will look for the eigenvectors of matrix (8). As usual, we will take advantage of the special form of this matrix: if we denote by $\mathbf{M}_\lambda$ the matrix $\mathbf{M}-\lambda\mathbf{I}_4$, then

$$\mathbf{M}_\lambda=\begin{bmatrix}\mathbf{A}_\lambda & \mathbf{I}_2\\ \mathbf{B} & \mathbf{A}_\lambda\end{bmatrix}, \qquad \mathbf{A}_\lambda=\begin{bmatrix}-\lambda & 1\\ -1 & -\lambda\end{bmatrix}, \qquad \mathbf{B}=\begin{bmatrix}2c_2 & 0\\ 0 & -c_2\end{bmatrix}.$$

Now, the kernel of $\mathbf{M}_\lambda$ can be found as follows: denoting as $\begin{bmatrix}\mathbf{w}_1^{\mathrm{T}} & \mathbf{w}_2^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$ the elements of the kernel, we start solving $(\mathbf{B}-\mathbf{A}^2)\mathbf{w}_1=\mathbf{0}$ and then $\mathbf{w}_2=-\mathbf{A}\mathbf{w}_1$. Thus, the eigenvectors of $\mathbf{M}$ are given by $\left[2\lambda,\quad \lambda^2-2c_2-1,\quad \lambda^2+2c_2+1,\quad \lambda^3+\left(1-2c_2\right)\lambda\right]^{\mathrm{T}}$, where λ denotes the eigenvalue.

Let us consider now the eigenvectors related to ω_1. From $p(\lambda)=0$, we obtain that ω_1 verifies

$$\omega_1^4-\left(2-c_2\right)\omega_1^2+\left(1+c_2-2c_2^2\right)=0.$$

We also apply $\lambda=\sqrt{-1}\omega_1$ to the expression of the eigenvector and, separating real and imaginary parts as $\mathbf{u}_{\omega_1}+\sqrt{-1}\mathbf{v}_{\omega_1}$ we obtain

$$\mathbf{u}_{\omega_1}=\left(0,-\omega_1^2-2c_2-1,-\omega_1^2+2c_2+1,0\right)^{\mathrm{T}}$$

$$\mathbf{v}_{\omega_1}=\left(2\omega_1,0,0-\omega_1^3+\left(1-2c_2\right)\omega_1\right)^{\mathrm{T}}.$$

Now, let us consider the eigenvalues related to $\pm\lambda_1$,

$$\mathbf{u}_{+\lambda_1}=\left(2\lambda,\lambda^2-2c_2-1,\lambda^2+2c_2+1,\lambda^3+\left(1-2c_2\right)\lambda\right)^{\mathrm{T}},$$
$$\mathbf{v}_{-\lambda_1}=\left(-2\lambda,\lambda^2-2c_2-1,\lambda^2+2c_2+1,-\lambda^3-\left(1-2c_2\right)\lambda\right)^{\mathrm{T}}$$

We consider, initially, the change of variables $\mathbf{C}=\left(\mathbf{u}_{+\lambda_1},\mathbf{u}_{\omega_1},\mathbf{v}_{-\lambda_1},\mathbf{v}_{\omega_1}\right)$. To know whether this matrix is symplectic or not, we check $\mathbf{C}^{\mathrm{T}}\mathbf{J}\mathbf{C}=\mathbf{J}$. It is a tedious computation to see that

$$\mathbf{C}^{\mathrm{T}}\mathbf{J}\mathbf{C}=\begin{bmatrix}\mathbf{0} & \mathbf{D}\\ -\mathbf{D} & \mathbf{0}\end{bmatrix},\qquad \mathbf{D}=\begin{bmatrix}d_{\lambda_1} & 0\\ 0 & d_{\omega_1}\end{bmatrix}.$$

This implies that we need to apply some scaling on the columns of $\mathbf{C}$ in order to have a symplectic change. The scaling is given by the factors

$$d_{\lambda_1}=2\lambda_1\left(\left(4+3c_2\right)\lambda_1^2+4+5c_2-6c_2^2\right),\qquad d_{\omega_1}=\omega_1\left(\left(4+3c_2\right)\omega_1^2-4-5c_2+6c_2^2\right).$$

Thus, we define $s_1=\sqrt{d_{\lambda_1}}$ and $s_2=\sqrt{d_{\omega_1}}$. As we want the change to be real, we have to require $d_{\lambda_1}>0$ and $d_{\omega_1}>0$. It is not difficult to check that this condition is satisfied for $0<\mu\le\frac{1}{2}$ in all the points $L_{1,2,3}$, if $\lambda_1>0$ and $\omega_1>0$.

To obtain the final change, we have to take into account the vertical direction $\left(z,p_z\right)$: to put it into real normal form we use the substitution

$$z \mapsto \frac{1}{\sqrt{\omega_2}} z, \qquad p_z \mapsto \sqrt{\omega_2} p_z .$$

This implies that the final change is given by the symplectic matrix

$$\mathbf{C} = \begin{bmatrix} \frac{2\lambda}{s_1} & 0 & 0 & -\frac{2\lambda}{s_1} & \frac{2\omega_1}{s_2} & 0 \\ \frac{\lambda^2-2c_2-1}{s_1} & \frac{-\omega_1^2-2c_2-1}{s_2} & 0 & \frac{\lambda^2-2c_2-1}{s_1} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\omega_2}} & 0 & 0 & 0 \\ \frac{\lambda^2+2c_2+1}{s_1} & \frac{-\omega_1^2+2c_2+1}{s_2} & 0 & \frac{\lambda^2+2c_2+1}{s_1} & 0 & 0 \\ \frac{\lambda^3+(1-2c_2)\lambda}{s_1} & 0 & 0 & \frac{-\lambda^3-(1-2c_2)\lambda}{s_1} & \frac{-\omega_1^3+(1-2c_2)\omega_1}{s_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\omega_2} \end{bmatrix} \tag{9}$$

that casts Hamiltonian (6) into its real normal form,

$$H_2 = \lambda_1 x p_x + \frac{\omega_1}{2}\left(y^2 + p_y^2\right) + \frac{\omega_2}{2}\left(z^2 + p_z^2\right) \tag{10}$$

where, for simplicity, we have kept the same name for the variables. Later on we will use a complex normal form for H_1 because it will simplify the computations. This complexification is given by

$$\begin{aligned} x &= q_1, & y &= \frac{q_2 + \sqrt{-1} p_2}{\sqrt{2}}, & z &= \frac{q_3 + \sqrt{-1} p_3}{\sqrt{2}}, \\ p_x &= p_1, & p_y &= \frac{\sqrt{-1} q_2 + p_2}{\sqrt{2}}, & p_z &= \frac{\sqrt{-1} q_3 + p_3}{\sqrt{2}}, \end{aligned} \tag{11}$$

and it puts (10) into its complex normal form,

$$H_2 = \lambda_1 q_1 p_1 + \sqrt{-1}\omega_1 q_2 p_2 + \sqrt{-1}\omega_2 q_3 p_3, \tag{12}$$

being λ_1, ω_1 and ω_2 real (and positive) numbers.

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[30] F. Gabern, W.S. Koon, J.E. Marsden, and D.J. Scheeres (2006). Binary asteroid observation orbits from a global dynamical perspective. *SIAM J. Appl. Dyn. Syst.*, 5(2):252-279. [The paper studies the dynamics of a particle near the triangular equilibrium points of a 3D restricted full three-body problem, in which the two primaries are an ellipsoid and a sphere. The study is based on the use of a frequency analysis method to detect different kinds of motion in the regions considered].

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[33] G. Gómez, À. Jorba, J. Masdemont, and C. Simó (2001). Study refinement of semi-analytical Halo orbit theory. ESOC contract 8625/89/D/MD(SC), final report, European Space Agency, 1991. Reprinted as *Dynamics and mission design near libration points. Vol. III, Advanced methods for collinear points*, volume 4 of World Scientific Monograph Series in Mathematics. [This text develops a number of advanced methods for the study of the collinear libration points. This includes the use of invariant manifolds (stable, unstable and center) for the design and control of space missions].

[34] G. Gómez, À. Jorba, J. Masdemont, and C. Simó (1993). Study of Poincaré maps for orbits near Lagrangian points. ESOC contract 9711/91/D/IM(SC), final report, European Space Agency. Reprinted as *Dynamics and mission design near libration points. Vol. IV, Advanced methods for triangular points*, volume 5 of World Scientific Monograph Series in Mathematics, 2001. [This text explores, analyses and computes the kinds of motion that appear near the triangular libration points of the Earth-Moon system, looking for bounded orbits to be used in space missions. The mission analysis for some of these orbits is also discussed].

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[36] G. Gómez, J. Llibre, R. Martínez, and C. Simó (1985). Station keeping of libration point orbits. ESOC contract 5648/83/D/JS(SC), final report, European Space Agency. Reprinted as *Dynamics and mission design near libration points. Vol. I, Fundamentals: the case of collinear libration points*, Volume 2 of World Scientific Monograph Series in Mathematics, 2001. [This is a pioneering work on the use of dynamical systems tools to design space missions. More concretely, it uses invariant manifolds for the transfer and control of a spacecraft around a Halo orbit of the Earth-Sun system. It develops improved models for the dynamics taking into account the main perturbations and it also develop numerical methods to deal with these models and to refine the results for the JPL model. These techniques produce a very accurate nominal orbit that is the key for a station keeping that requires a low amount of fuel].

[37] G. Gómez, J. Llibre, R. Martínez, and C. Simó (1987). Study on orbits near the triangular libration points in the perturbed Restricted Three-Body Problem. ESOC contract 6139/84/D/JS(SC), final report, European Space Agency. Reprinted as *Dynamics and mission design near libration points*. Vol. II, Fundamentals: the case of triangular libration points, volume 3 of World Scientific Monograph Series in Mathematics, 2001. [This text focuses on the dynamics near the triangular libration points. It develops a hierarchy of models for the Earth-Moon case, and uses them to obtain bounded orbits near the triangular

points. These orbits can be suitable for some space missions].

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[40] G. Gómez, J.J. Masdemont, and C. Simó (1998). Quasihalo orbits associated with libration points. *J. Astronaut. Sci*., 46(2):135-176. [This paper deals with two dimensional tori around Halo orbits. It also contains a discussion of the practical convergence of the Lindstedt-Poincaré procedure and a numerical refinement of some of these orbits for the JPL model].

[41] G. Gómez and J.M. Mondelo (2001). The dynamics around the collinear equilibrium points of the RTBP. *Phys. D*, 157(4):283-321. [This paper is devoted to the analysis of an extended neighborhood of the collinear equilibrium points of the RTBP. The analysis is done using numerical tools for the determination of periodic orbits and invariant 2D tori].

[42] J. Henrard and J.F. Navarro (2004). Families of periodic orbits emanating from homoclinic orbits in the restricted problem of three bodies. *Celestial Mech*., 89(3):285-304. [The paper presents a numerical study of families of periodic orbits associated with homoclinic orbits of the triangular points of the planar RTBP when the mass parameter is larger than the Routh value].

[43] À. Jorba (1999). A methodology for the numerical computation of normal forms, centre manifolds and first integrals of Hamiltonian systems. *Exp. Math.*, 8(2):155-195. This paper describes the effective computation of normal forms, center manifolds and first integrals near the equilibrium points of the RTBP. The software is available from the webpage of the author].

[44] À. Jorba (2000). A numerical study on the existence of stable motions near the triangular points of the real Earth-Moon system. *Astron. Astrophys.*, 364(1):327-338. [It is known that the neighborhood of the triangular points of the real Earth-Moon system is unstable, mainly due to the effect of the Sun. This paper studies the existence of stable orbits using first the Bicircular model. Then, a simulation using the full Solar system (by means of the JPL ephemeris) shows that some of these orbits look stable for, at least, 1000 years].

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[46] À. Jorba and C. Simó (1994). Effective stability for periodically perturbed Hamiltonian systems. In J. Seimenis, editor, *Hamiltonian Mechanics: Integrability and Chaotic Behaviour*, volume 331 of NATO *Adv. Sci. Inst. Ser. B Phys.*, pages 245-252. Held in Torun, Polland, 28 June-2 July 1993. Plenum, New York. [This note deals with the computation of the (resonant) normal form of the Hamiltonian of the elliptic 3D RTBP at $L_{4,5}$, for the Sun-Jupiter case. The estimates on the size of the remainders are used to bound the diffusion near the triangular points.

[47] À. Jorba and C. Simó (1996). On quasiperiodic perturbations of elliptic equilibrium points. *SIAM J. Math. Anal*., 27(6):1704-1737. [The paper considers an analytic quasi-periodic perturbation of an analytic autonomous ODE near an equilibrium point. It is shown that there exists a small quasi-periodic solution with the same frequencies as the perturbation, for most values of the perturbing parameter].

[48] À. Jorba and J. Villanueva (1997a). On the normal behaviour of partially elliptic lower dimensional tori of Hamiltonian systems. *Nonlinearity*, 10:783-822. [This paper focuses on the elliptic directions of lower dimensional tori $\mathcal{T}$ of an autonomous Hamiltonian system. Under generic conditions, it is shown that each set of elliptic directions gives rise to a Cantor family of (higher) dimensional tori. This part can be seen as an extension to the Lyapunov centre theorem. Moreover, if $\mathcal{T}$ is completely elliptic, then it is shown that the time to move away from it is exponentially large with the initial distance to $\mathcal{T}$].

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[50] À. Jorba and J. Villanueva (1998). Numerical computation of normal forms around some periodic orbits of the Restricted Three Body Problem. *Phys. D*, 114(3-4):197-229. [This paper studies the stability of the vertical family of periodic orbits of the 3D RTBP, for a value of the mass parameter larger than the Routh value. The dynamics near the periodic orbit is described by means of a normal form computation, and the stability is derived from suitable bounds on the remainder].

[51] A.N. Kolmogorov (1954). On the persistence of conditionally periodic motions under a small change of the Hamilton function. *Dokl. Acad. Nauk. SSSR*, 98(4):527-530. [This is the first statement of the celebrated KAM theorem. The ideas explained there were a breakthrough in Hamiltonian mechanics and perturbation theory, and gave rise to what we know as KAM theory].

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[53] W.S. Koon, J.E. Marsden, S.D. Ross, and M.W. Lo (2002). Constructing a low energy transfer between Jovian moons. In *Celestial mechanics* (*Evanston, IL, 1999*), volume 292 of *Contemp. Math*., pages 129-145. Amer. Math. Soc., Providence, RI. [The paper is devoted to the design of a space mission to orbit Jupiter's moon Europa. The procedure developed takes advantage of the natural orbital dynamics when it is approached by two coupled RTBP. The transit between both RTBPs is done using some sort of heteroclinic connection between the Lyapunov libration point orbits of these two systems].

[54] Ch. Lhotka, C. Efthymiopoulos, and R. Dvorak (2008). Nekhoroshev stability at L_4 or L_5 in the elliptic-restricted three-body problem -application to Trojan asteroids. *Mon. Not. R. Astron. Soc.*, 384(3):1165-1177. [This work studies the Nekhoroshev stability in the case of the planar elliptic RTBP. To this end, the authors introduce an explicit symplectic mapping model obtained via Hadjidemetriou's method. The diffusion estimates are obtained from approximate integrals of motion obtained from a normalizing process of this map].

[55] C. Marchal (1980). The quasi integrals. *Celestial Mech*., 21:183-191. [This paper introduces the construction of formal first integrals to describe the dynamics of a nearly integrable Hamiltonian system].

[56] C. Marchal (2009). Long term evolution of quasi-circular Trojan orbits. *Celestial Mech.*, 104(1-2):53-67. [This paper deals with the stability of orbits near the triangular points of the 3D RTBP, for small values of the mass parameter. The paper discusses the long term variation of the orbital elements of trajectories close to the Lagrangian points].

[57] A.P. Markeev (1969). On the stability of the triangular libration points in the circular bounded three-body problem. *J. Appl. Math. Mech.*, 33:105-110. [This paper deals with the Lyapunov stability of the triangular points in the planar RTBP, by means of the KAM theorem. It is shown that the triangular libration points are stable for all ratios of the masses in the stability range, with the exception of certain specific ratios for which they are unstable].

[58] J.E. Marsden and S.D. Ross (2006). New methods in celestial mechanics and mission design. *Bull. Amer. Math. Soc. (N.S.),* 43(1):43-73. [This paper focuses on the influence of dynamical systems in the design of complex space trajectories. As examples, the authors mention the Genesis mission, the Lunar Gateway Station concept or a Jovian Tour about the Moons of Jupiter].

[59] C.R. McInnes (1999). *Solar Sailing: Technology, Dynamics and Mission Applications*. Springer-Praxis. [This book introduces the basic ideas on Solar sailing, including a discussion about the equilibria in the RTBP with solar radiation pressure].

[60] K.R. Meyer and G.R. Hall (1992). *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*. Springer, New York. [This book presents the fundamentals of Hamiltonian dynamics and their application to Celestial Mechanics].

[61] A. Morbidelli and A. Giorgilli (1995). On a connection between KAM and Nekhoroshev theorem. *Phys. D*, 86(3):514-516. [This paper describes a stability result for nearly integrable Hamiltonian systems with a convex integrable Hamiltonian, showing the relation between KAM tori (perpetual stability) and the region of Nekhoroshev stability: when the stability time grows to infinity, the stability region shrinks to the Cantor set of KAM tori].

[62] J. Moser (1962). On invariant curves of area-preserving mappings of an annulus. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, 2:1-20. [This paper shows the persistence of invariant curves in area preserving twist" maps. This is one of the pioneering works in KAM theory].

[63] N.N. Nekhoroshev (1977). An exponential estimate of the time of stability of nearly-integrable Hamiltonian systems. *Russian Math. Surveys*, 32:1-65. [This is a pioneering paper on the stability of nearly integrable Hamiltonian systems. While the KAM theorem gives perpetual stability for a Cantor set of initial conditions, this paper gives a very long (but finite) bound on the diffusion time valid on an open set of initial conditions].

[64] M. Ollé, J.R. Pacha, and J. Villanueva (2004). Motion close to the Hopf bifurcation of the vertical family of periodic orbits of L_4. *Celestial Mech.*, 90(1-2):89-109. [The paper deals with the dynamics close to the Lyapunov vertical family of periodic orbits of the triangular point L_4 in the 3D RTBP, when the mass parameter is greater than (but close to) the Routh critical value].

[65] R. Pérez-Marco (2003). Convergence or generic divergence of the Birkhoff normal form. *Ann. of Math. (2)*, 157(2):557-574. [The author proves that the Birkhoff normal form of Hamiltonian flows at a nonresonant singular point with given quadratic part is always convergent or generically divergent. The same result is proved for the normalization mapping and any formal first integral].

[66] A.D. Pinotsis (2010). In finite Feigenbaum sequences and spirals in the vicinity of the Lagrangian periodic solutions. *Celestial Mech*., 108(2):187-202. [The paper presents a study of certain families of nonsymmetric orbits in the planar RTBP with equal masses. In particular, it focuses on the description of the main characteristics of the families and the changes of the stability of the orbits and the bifurcations].

[67] D.L. Richardson (1980). Analytic construction of periodic orbits about the collinear points. *Celestial Mech*., 22(3):241-253. [The paper deals with the computation of third-order approximations to Halo orbits about the collinear points of the RTBP. The solution is constructed using the method of successive approximations in conjunction with a technique similar to the Lindstedt-Poincaré method. The theory is applied to the Sun-Earth system].

[68] D.L. Richardson (1980). A note on a Lagrangian formulation for motion about the collinear points. *Celestial Mech.*, 22(3):231-236. [The paper discusses the Lagrangian formulation for the 3D motion of a satellite in the vicinity of the collinear points of the RTBP. It is shown that the equations for the motion can be developed into highly compact expressions].

[69] P. Robutel and J. Bodossian (2009). The resonant structure of Jupiter's Trojan asteroids-II. What happens for different configurations of the planetary system. *Mon. Not. R. Astron*. Soc., 399:69-87. [This paper discusses Trojan motion for generic planetary systems, with a focus on the effect that a planetary migration can have on these motions. This method is used to study the global dynamics of the Jovian Trojan swarms when Saturn migrates outwards].

[70] P. Robutel and F. Gabern (2006). The resonant structure of Jupiter's Trojan asteroids-I. Long-term stability and diffusion. *Mon. Not. R. Astron. Soc*., 372:1463-1482. [This paper studies the dynamics of the jovian Trojan asteroids by means of the frequency analysis. The main resonances are identified and discussed. This global view of the dynamics is related with the observed Trojans].

[71] P. Robutel, F. Gabern, and A. Jorba (2005). The observed Trojans and the global dynamics around the Lagrangian points of the Sun-Jupiter system. *Celestial Mech.*, 92(1-3):53-69. [This paper deals with the dynamical structure of the Sun-Jupiter L_4 tadpole region. The results are based on long-time simulations in the Sun-Jupiter-Saturn system. The results are connected with the observed Trojans and the resonances corresponding to some real bodies are identified].

[72] P. Robutel and J. Souchay (2010). An introduction to the dynamics of Trojan asteroids. In J. Souchay and R. Dvorak, editors, *Dynamics of Small Solar System Bodies and Exoplanets*, volume 790 of *Lect. Notes Phys*., pages 195-227. Springer. [This is a survey about the motion near the triangular points of the RTBP. The authors discuss several aspects of the real motion of Trojan asteroids, including both

theoretical and applied results].

[73] B. Sicardy (2010). Stability of the triangular Lagrange points beyond Gascheau's value. *Celestial Mech.*, 107(1-2):145-155. [The paper deals with the stability of $L_{4,5}$ of the planar RTBP for a mass parameter μ slightly larger than the Routh (Gascheau) value. It is shown that if $\mu < 0.039$, $L_{4,5}$ still present some kind of stability. Moreover it is also shown that there exists a family of stable periodic orbits presenting a Feigenbaum cascade (period doublings), μ leading to disappearance into chaos at a value $\mu = 0.0463004$].

[74] C. L. Siegel and J. K. Moser (1971). *Lectures on Celestial Mechanics*, volume 187 of *Grundlehren Math. Wiss*. Springer, New York. [This is a classical book on celestial mechanics, including some material from Hamiltonian dynamical systems and, in particular, from KAM theory].

[75] J. Sijbrand (1985). Properties of center manifolds. *Trans. Amer. Math. Soc.*, 289(2):431-469. [This paper is a basic survey about center manifolds, discussing the main issues about existence, uniqueness and differentiability].

[76] C. Simó (1989). Estabilitat de sistemes Hamiltonians. *Mem. Real Acad. Cienc. Artes Barcelona*, 48(7):303-348. [This paper studies the stability of an elliptic equilibrium point of an autonomous Hamiltonian system, focusing on the $L_{4,5}$ points of the spatial RTBP. To enlarge the stability region obtained by means of a complete normal form procedure, the author uses a seminormal form that accounts for a relevant resonance. The technique is applied to the Sun-Jupiter RTBP].

[77] C. Simó, G. Gómez, À. Jorba, and J. Masdemont (1995). The Bicircular model near the triangular libration points of the RTBP. In A.E. Roy and B.A. Steves, editors, *From Newton to Chaos*, pages 343-370, New York. Plenum Press. [This is a study of the Bicircular model near the Lagrangian points of the Earth-Moon system. It is shown that the region near $L_{4,5}$ is unstable but that there exists a stability zone at some distance of the Lagrangian points].

[78] V. Szebehely (1967). *Theory of Orbits*. Academic Press. [This is a classical textbook on the Restricted Three-Body Problem. It contains basic but fundamental material on this problem].

[79] A. Vanderbauwhede (1989). Centre manifolds, normal forms and elementary bifurcations. In *Dynamics reported, Vol. 2*, volume 2 of *Dynam. Report. Ser. Dynam. Systems Appl.*, pages 89-169. Wiley, Chichester. [The paper presents a proof of the center manifold theorem, discussing its existence, uniqueness and differentiability. Then, normal forms of vector fields near singular points are discussed. The last part deals with codimension 1 bifurcations].

[80] C.G. Zagouras and P.G. Kazantzis (1979). Three-dimensional periodic oscillations generating from plane periodic ones around the collinear Lagrangian points. Astrophys. *Space Sci.*, 61:389-409. [This paper presents a numerical computation of the family of Halo orbits for the Sun-Jupiter RTBP].

Biographical Sketch

Àngel Jorba (born in 1963 in Barcelona, Spain) received his PhD from the University of Barcelona in 1991 under the supervision of Carles Simó. He has been associate professor at the Polytechnic University of Catalonia and he is currently Professor of Applied Mathematics at the University of Barcelona. He is a member of the editorial board of *Discrete and Continuous Dynamical Systems - Series B* since 2001, and he has served as coordinator of the Spanish network of dynamical systems (DANCE) from 2006 to 2010. His research interests include celestial mechanics and astrodynamics, with a particular interest in the analysis of space missions. He is also interested on the occurrence of quasi-periodic motions in dynamical systems, and in the development of numerical and semi-analytical tools to deal with the application of dynamical systems to real situations.