MATHEMATICAL MODELS AND CONTROL OF CATASTROPHIC PROCESSES

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Summary

The basic notions of Singularity Theory of differentiable mappings and the Bifurcation theory of dynamical systems are described.

They form the foundation of Mathematical Catastrophe Theory, which is essentially a qualitative analysis of complicated systems depending on parameters, such as life supporting systems in real life.

We demonstrate some simple examples of systems wherein the choice of the control parameter makes it possible to avoid abrupt (catastrophic) behavior.

The aim of the chapter is to emphasize that the general methodology of these mathematical theories are important in investigations of the specific complicated systems, even at the stage of creating adequate models.

1. Introduction

In mathematics sudden and abrupt changes in the response of a system to a smooth change in external conditions are termed as *catastrophes*.

The mathematical catastrophe models exhibit certain common traits of the most diverse phenomena of jump change in a system behavior. Mostly catastrophe theory is a collection of the applications and general ideas of Singularity Theory and Bifurcation Theory of dynamical systems.

The main object studied by these theories is a system which continuously depends on its parameters. Any mathematical model of a real system (for instance, physical ecological or economical) determines a set of numerical parameters, whose values reflects important features of the system.

In a relatively simple case (which might be called a *stationary system*) *the state of a system* is described by a point from a certain subset of the space of all possible values of parameters. This subset can have a complicated structure and singularities. Here we may well rely on the singularity theory of differentiable mapping.

In other cases, for *evolution processes*, time is one of the parameters and a *system* is modeled by a dynamical system, governed by certain system of differential equations or by relations of more complicated nature.

Even the simplest general mathematical conclusions on the properties of such systems can often help in investigation of specified complicated modeling problems.

The basic idea of the catastrophe theory according to E.C.Zeeman, is the following:

Assume that the parameters defining the states of the system are separated into two groups: *internal* and *external.* It is assumed that there exists dependence among the parameters. However, the values of the internal parameters are not uniquely determined

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by the values of the external ones. Geometrically the states of the system are described by the points in the product of the spaces of internal and external parameters. The meaning of the dependence is that this point (the current state of the system) always lies in some subset of the product space. In the simplest case one may assume that this subset is a smooth submanifold in general position in the product space and its dimension is equal to the dimension of the space of external parameters. The projection of this submainfold onto the space of external parameters is not generically one-to-one. In other words, even in this simplest model the internal parameters do not depend smoothly on the external ones: Under small changes of external parameters our system may jump from one visible state (i.e. from a point in the internal parameter space) to another one.

Singularity Theory yields information on the critical points and critical values of such mappings. Analytical, geometrical and topological methods often ensure the appearance of certain types of critical sets without ambiguity.

Since smooth mappings are found everywhere, their singularities must be everywhere also, and since Singularity Theory gives significant information on the singularities of generic mappings, one can try to use this information to study a lot of diverse phenomena and processes in all areas of science.

The mathematical description of the world depends on a delicate interplay between continuous and discontinuous (discrete) phenomena. The latters are perceived first. "Functions, just like living beings are characterized by their singularities", as P. Montel proclaimed.

The modern singularity theory began in the 1950s by the works of H. Whitney. Also in the early 1930s A.A Andropov started the theory of the bifurcations of dynamical systems. However similar ideas and objects go up to classics: Hamilton, Monge, Cayley, Poincaré. Now there exist thousands of publications on the development of their results and on various applications. The singularity theory is now one of the central areas of mathematics, where the most abstract parts (differential and algebraic geometry and topology, group theory, the theory of complex spaces) come together with the most applied ones (stability of motion of dynamical systems, bifurcation of equilibrium states, optics, optimal control).

Its methods were applied to various branches of knowledge, for example, to heat beats modeling, to geometrical and physical optics, to embryology, linguistics, economics, hydrodynamics, geology, computer vision, elasticity theory, stability of ships, etc.

Certain features of catastrophes in natural, technological, social and other processes might be understood using methods of Singularity Theory and Bifurcation Theory.

In Section 2 the general methods and simple examples of catastrophe theory are outlined.

That is, in Subsection 2.1 we describe the traditional example of Zeeman's "catastrophe mashine", which exhibits loss of stability and jump–like dynamics in mechanical

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systems with rigid and elastic elements. Also the bifurcation of steady state positions of loaded elastic beam is analyzed. These two models provide the simplest examples of the real catastrophe which may occur in the building and construction technology.

In Subsection 2.2 the simplest resource exploitation model (based on natural law of resource growth) is considered. We show that the catastrophic vanishing of the resource due to the intensification of extraction can be avoided by the manipulation of a feedback control parameter. In this simplest model of the “control of the ecological catastrophe” the average production might be maintained at the optimal level.

In the following sections recent developments in these theories and domains of their application are described.

Singularities of functions described in Subsection 3.1 provide mathematical models of typical catastrophes in systems with several external parameters. The knowledge of their bifurcation diagrams or discriminant sets (from Subsection 3.2) is necessary to control behavior of the system smoothly.

The propagation of various catastrophic disturbances (e.g. shock waves, emanations, epidemic or a flame) in certain media has many common features modeled by the theory of wavefronts and caustics (outlined in Subsection 3.3). This covers a vaste area of applications of singularity theory, including Y. Zeldovich’s model of the catastrophic formation of the Universe. An interesting new approach to the problem of the choice of a good decision is described in Subsection 3.4. Political and social sciences provide enough examples when the wrong decision leads to real catastrophe.

Different models of shock fronts are described in Subsection 3.5.

The implementation of the control parameters into a system does not lead automatically to the perishing of the catastrophes.

To avoid catastrophe in the behavior of a system one has to know the typical singularities of the control systems themselves. This is the subject of Section 4. Here we again meet the simplest models based on the singularities of families of functions (conflicts sets) and bifurcations of singular points of vector and direction fields (Subsection 4.2). We have to emphasize that to suggest a detailed and adequate mathematical model for real life supporting systems and to prove their consistency is the subject of the specific sciences going beyond the aims of the present article.

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Bibliography

Arnold V.I. (1990). *Singularities of Caustics and Wavefronts*, Dorderecht, The Netherlands: Kluwer Academic Publ., 259pp. [The survey of basic and recent notions and methods of singularities in sympletic and contact spaces. It describes numerous interactions of these theories with wave theory, variational problems etc. In particular it contains the theory and classification of boundary singularities and its applications to varitional problems with an obstacle.]

Arnold V.I. (1992). *Catastrophe Theory*, Third Revised Edition, Berlin a.o: Springer Verlag, [A short, critical review of catastrophe theory which is oriented to non-mathematicians. It provides a useful introduction to the subject and contains an extensive bibliography.]

Arnold V.I. (1994). Catastrophe Theory, *Encyclopedia of Mathematical Sciences, vol. 5, Dynamical Systems*, Berlin a.o:

Arnold V.I., Goryunov V.V., Lyashko O.V. and Vasil'ev V.A. (1993) Singularites I, II, Encyclopedia of Math Sciences, Vol. 6, 39, Dynamical Systems VI and VIII, Berlin a, o.: Springer –Verlag. 245-235 pp. [A collection of contemporary results of singularity theory of diffentiable mappings and various applications in geometry and anlaysis. It contains in particular an introduction to Picard-Lefschetz theories and various their applications to integral geometry and mathematical physics.]

Davydov A.A. (1994) *Qualitative Theory of Control Systems*,. Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1994. ISBN: 0-8218-4590-X. [Applications of Singularity Theory to the study of controllability properties of typical control systems.]

Glimore R. (1981). *Catastrophe theory for scientists and engineers*, New York, USA: Willey–Interscience .666pp. [A detailed description of applications of Singularity theory to problems in the elasticity theory and mechanics.]

Landau L. D. and Lifshits E..M.(1958). *Statistical physics*, Pergamon Press. 484pp. [Classical monograph (and at the same time a highest level textbook) written sometime without strict mathematical preciseness. Contains general principals of Thermodynamics (where the singularities of Legendre transformation play an important role). In particular the isotherms of the Van der Waals' gas exhibit the standard metamorphoses of catastrophe theory.]

Smale S. (2000). Mathematical problems for the 21 th century. *Mathematics: Frontiers and Perspectives*. (V. Arnold, M. Atiyah, P. Lax and B. Mazur, eds.) Providence RI, USA: American Mathematical Society. [The annotated list of problems, which one of the great contemporary mathematicians (whose ideas were of great influence for the catastrophe theory) considers among the most important for the development of mathematics in 21 th century. Contains in particular a sketch and reference list of works of Smale, M. Shub and others on the path–following method in applied mathematics.]

Springer –Verlag. [The volume contains two survey articles. The first one is a detailed but brief description of principal results in Bifurcation Theory of Dynamical Systems. The second part (Catastrophe Theory) is devoted to short description of recent applications of singularity Theory. Various interactions of Catastrophe Theory with classical works of mathematicians and physicists are traced. The volume contains also extensive bibliography and historical remarks on the development of the Catastrophe Theory.]

Vassiliev V. (1994) *Complements of Discriminants of Smooth Maps: Topology and Applications*. *Revised ed.* Providence RI, USA: American Mathematical Society, ser. Transl. of Mathematical Monographs. [Introduction to the topological study of discriminant sets in functional spaces. Describes In particular the application of theory to the complexity of the problems of choice]

Zakalyukin V.M. (1998). Concave Darboux Theorem, *C.R. Acad. Sci Paris, ser 1*, **327**,633-638 pp. [A first example of the use of symplectic and contact geometries in mathematical economics]

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