

METHODS OF POTENTIAL THEORY

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Summary

The Laplace equation $\Delta u = 0$ or $\nabla^2 u = 0$ is one of the basic classical equations of mathematical physics. Its solution is represented as the integral of the product of some function (potential density) and the fundamental solution of the Laplace equation. An integral of this kind is said to be a potential integral. In the 3D case the fundamental solution of the Laplace equation is the function $1/r$ and in the 2D case- the function $\ln 1/r$, where r is the distance between points. If we look for a solution of a boundary value problem in the form of a potential, then for potential density we obtain the Fredholm integral equation, where the integration is performed over the boundary of a given domain. The potential theory can be naturally extended to more complicated elliptic equations and other equations of mathematical physics.

1. Introduction

The notion of Newton's potential was first introduced at the end of the 18th century by P.Laplace and J.Lagrange and then by L. Euler for problems of hydrodynamics. The notion of a potential being considered as a function whose gradient is a vector field is due to Gauss. The properties of the simple layer potential were first studied by Coulomb and Poisson, the great contribution to the development of the potential theory was made by Green. Nowadays the potential theory is an actively developed tool for studying and solving problems in different fields of mathematical physics.

Let $\mathbf{F} = \sum_{i=1}^3 F_i \mathbf{e}_i$ be a given vector field, where $F_i = F_i(x, y, z)$ are the component of the vector \mathbf{F} applied at the point (x, y, z) , \mathbf{e}_i are the basic vectors of the orthogonal

coordinate system; let $u(x, y, z)$ be a scalar function (scalar field). A scalar field $u(x, y, z)$ whose gradient equals \mathbf{F} : $\text{grad } u = \nabla u = (\partial u / \partial x, \partial u / \partial y, \partial u / \partial z) = \mathbf{F}$, is called the potential of a vector field \mathbf{F} . So knowledge of a potential function (potential) allows one to calculate acting forces. Many problems of electromagnetism, hydrodynamics, acoustics, heat conductivity and diffusion are reduced to boundary value problems for elliptic equations. The simplest and important examples of such equations are the Laplace equation $\Delta u = 0$ and the Poisson equation $\Delta u = f$. Here Δ is the Laplace

operator $\Delta u = \sum_{i=1}^3 \partial^2 u_i / \partial x_i^2$. The fundamental solutions of the Laplace equation being

equal to $(4\pi r)^{-1}$ in the 3D case and to $(2\pi)^{-1} \ln(1/r)$ in the 2D case play key role in the methods of the potential theory. On the basis of these solutions a potential is constructed as the integral of the product of some function (potential density) and a fundamental solution (or its derivative). Depending on an integration domain and on the use of a fundamental solution or its normal derivative, the volume potentials and the simple and double layer potentials are distinguished. If we look for a potential (a solution of the corresponding elliptic equation) in the form of the integral of density, then we obtain an integral equation for the unknown density. Since the solution can be expressed in terms of different potentials, the preferred choice of a potential is that which yields the simplest integral equation. Thus, to obtain the Fredholm equation of the second kind, the Dirichlet problem should be solved with the help of the double layer potential and the Neumann problem should be solved with the simple layer potential. Below we consider the potentials for the Laplace and Helmholtz equations and the wave and heat conductivity equations being the basic types of equations of mathematical physics that arise in energetics, ecology, the theory of electricity, atmosphere and ocean.

2. Fundamentals of the Potential Theory

2.1. Some Elements from Calculus

2.1.1. Basic Orthogonal Coordinates

Given a system of three single-valued functions of three variables:

$$\left. \begin{aligned} x_1 &= \varphi_1(u_1, u_2, u_3), \\ x_2 &= \varphi_2(u_1, u_2, u_3), \\ x_3 &= \varphi_3(u_1, u_2, u_3). \end{aligned} \right\} \quad (1)$$

Suppose that to each set of values u_1, u_2, u_3 there corresponds a certain point M in the space with Cartesian coordinates x_1, x_2, x_3 . The quantities u_1, u_2, u_3 can be considered as curvilinear coordinates of the point M . They define a coordinate system which is said to be *curvilinear*. A system is called *orthogonal* if at each point the coordinate lines passing through this point mutually intersect at right angles. Let us consider two basic examples of curvilinear orthogonal coordinates.

1° *Cylindrical coordinates:*

$$x = r \cos \varphi, \quad u = r \sin \varphi, \quad z = z \quad (\varphi \in [0, 2\pi], r > 0).$$

Here instead of x_1, x_2, x_3 we have x, y, z and instead of $u_1, u_2, u_3 - r, \varphi, z$. In the 2D case being independent of z cylindrical coordinates are called polar coordinates.

2° *Spherical coordinates*

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta \quad (\theta \in [0, \pi], \varphi \in [0, 2\pi], r > 0).$$

2.1.2. Basic Differential Operations on a Vector Field

Let $\varphi = \varphi(u_1, u_2, u_3)$ be a scalar field, $\mathbf{F} = \mathbf{F}(u_1, u_2, u_3)$ be a vector field, $\mathbf{F} = \sum_{i=1}^3 F_i \mathbf{e}_i$. In the Cartesian rectangle coordinates the following operations are defined.

Gradient:

$$\text{grad } \varphi = \nabla \varphi = \sum_{i=1}^3 \partial_i \varphi \mathbf{e}_i;$$

Divergence:

$$\text{div } \mathbf{F} = (\nabla, \mathbf{F})_3 = \sum_{i=1}^3 \partial_i F_i;$$

Rotor (vorticity):

$$\text{rot } \mathbf{F} = [\nabla, \mathbf{F}] = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix};$$

The Laplace operator (Laplacian)

$$\Delta \varphi = \text{div grad } \varphi = \sum_{i=1}^3 \partial_i^2 \varphi,$$

where the designations $\partial_i = \partial / \partial u_i, \partial_i^2 = \partial^2 / \partial u_i^2, \nabla = (\partial_1, \partial_2, \partial_3)$ are introduced for the sake of convenience. In cylindrical coordinates the Laplace operator has the form

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \varphi^2} + \frac{\partial^2 v}{\partial z^2} \quad (2)$$

and in spherical coordinates

$$\Delta v = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \left(\sin \theta \frac{\partial v}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v}{\partial \varphi^2}. \quad (3)$$

2.1.3. Formulae from the Field Theory

Let u and v be two arbitrary functions with continuous partial derivatives up to the second order inclusive. Instead of $u = u(x, y, z)$, we write $u = u(A)$ where a point A has coordinates (x, y, z) . The distance between a point $A(x, y, z)$ and a point $P(\xi, \eta, \zeta)$ is defined by

$$r_{AP} = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}.$$

The symbols of differential operators on functions of A and P will be equipped with the sub-scripts A or P depending on whether the differentiation is performed with respect to x, y, z or ξ, η, ζ . For, example,

$$\Delta_A u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad \text{grad}_P u = \frac{\partial u}{\partial \xi} \mathbf{i} + \frac{\partial u}{\partial \eta} \mathbf{j} + \frac{\partial u}{\partial \zeta} \mathbf{k}.$$

The symbol $(\partial u / \partial n)_P$ denotes the derivative in the direction of the normal \mathbf{n} to a surface at a point P :

$$\left(\frac{\partial u}{\partial n} \right)_P = \frac{\partial u}{\partial \xi} \cos \alpha + \frac{\partial u}{\partial \eta} \cos \beta + \frac{\partial u}{\partial \zeta} \cos \gamma,$$

where $\cos \alpha, \cos \beta, \cos \gamma$ are the direction cosines of the outer normal \mathbf{n} . Recall the *Ostrogradskii-Gauss formula*

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV = \iint_S (P \cos \alpha + Q \cos \beta + R \cos \gamma) dS,$$

where the cosines are the direction cosines of an outer normal \mathbf{n} . Setting $P = u \partial_1 v$, $Q = u \partial_2 v$, $R = u \partial_3 v$, we arrive at *Green's first formula*

$$\iiint_V (\text{grad } u, \text{grad } v) dV + \iiint_V u \Delta v dV = \iint_S u \frac{\partial v}{\partial n} dS. \quad (4)$$

Let us interchange u and v in the formula (4) and subtract the obtained equality from (4). This yields *Green's second formula*:

$$\iiint_V \{u \Delta v - v \Delta u\} dV = \iint_S \left\{ u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right\} dS. \quad (5)$$

If $A \in V$, then we can not substitute $v = 1/r_{AP}$ at once into (5). Construct a small sphere with the centre at A . Applying Green's second formula (5) to the functions u and v outside the sphere and assuming that the radius of the sphere tends to zero, we obtain *Green's main integral formula*

$$\Omega \cdot u(A) = \iint_S \left\{ \frac{1}{r_{AP}} \frac{\partial v}{\partial n} - u(P) \frac{\partial(1/r_{AP})}{\partial n} \right\} dS_P - \iiint_V \frac{\Delta u(P)}{r_{AP}} dV_P. \quad (6)$$

Depending on the location of the point A , the coefficient Ω takes the values $\Omega = 4\pi, A \in V, \Omega = 2\pi, A \in \partial V, \Omega = 0, A \notin V$. Similarly, in the 2D case we denote some domain in the plane (x, y) , bounded by a smooth closed curve L (or by several curves), by D . Then for arbitrary functions u and v , which have continuous partial derivatives up to the second order inclusive, the following expressions take place:

$$\iint_D \left\{ \frac{\partial u}{\partial \xi} \frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial v}{\partial \eta} \right\} dS + \iint_D u \Delta v dS = \oint_L u \frac{\partial v}{\partial n} dl, \quad (7)$$

$$\iint_D \{u \Delta v - v \Delta u\} dS = \oint_L \left\{ u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right\} dl, \quad (8)$$

$$u(A) = \frac{1}{2\pi} \oint_L \left\{ \ln \frac{1}{r} \frac{\partial u}{\partial n} - u(P) \frac{\partial(\ln(1/r))}{\partial n} \right\} dl - \frac{1}{2\pi} \iint_D \Delta u \ln \frac{1}{r} dS, \quad (9)$$

where $\partial/\partial n$ is the differentiation operator in the direction of outer normal to L , $\Delta = \partial^2/\partial \xi^2 + \partial^2/\partial \eta^2, r = r_{AP}$ is the distance between the point A and a variable point P .

2.1.4. Basic Properties of Harmonic Functions

Functions, satisfying the Laplace equation $\Delta u = 0$ in a domain V , are called harmonic functions. For a harmonic function U the following properties hold.

$$1^\circ \cdot \iint_S \frac{\partial U}{\partial n} dS = 0,$$

i.e., *the integral of the normal derivative of a harmonic function over the boundary of a domain is equal to zero.*

2° · *The value of a harmonic function at any interior point of a domain is expressed in terms of the values of this function and its normal derivative at the boundary of the domain by the formula*

$$U(A) = \frac{1}{4\pi} \iint_S \left[\frac{1}{r} \frac{\partial U}{\partial n} - U \frac{\partial(1/r)}{\partial n} \right] dS.$$

3°. *The value of a harmonic function at the centre A of a sphere S_R of radius R is equal to the arithmetic mean of the values of this function at the surface of the sphere, i.e., to the integral of the function over the surface of the sphere, divided by the area of this surface:*

$$U(A) = \frac{1}{4\pi R^2} \iint_{S_R} U dS.$$

4°. *From 3° the maximum principle follows: a function, harmonic inside a domain and continuous up to its boundary, takes its maximum and minimum values at the boundary of the domain.*

2.2. Volume Mass or Charge potential

2.2.1. Newton's (Coulomb's) potential

Let V be some finite domain in R^3 , bounded by a piecewise smooth closed surface S . Let $\rho(P)$ be a continuous bounded function in V . Then

$$u(A) = \iiint_V \frac{\rho(P)}{r} dV \quad (10)$$

is called *the infinite mass potential or Newton's mass potential distributed over volume V with density ρ* . The function $u(A)$ can also be considered as Coulomb's potential of volume-distributed charges.

2.2.2. Properties of Newton's Potential

At any point A outside V the function $u(A)$ from (10) is continuous and differentiable with respect to x, y, z under the integral sign as much times as desired. In particular,

$$\text{grad } u(A) = \iiint_V \rho(P) \text{grad}(1/r) dV = - \iiint_V \rho(P) \frac{\mathbf{r}}{r^3} dV, \quad (11)$$

where \mathbf{r} is a radius-vector, $\mathbf{r} =$

$$\mathbf{r}_{AP} = (x - \xi)\mathbf{i} + (y - \eta)\mathbf{j} + (z - \zeta)\mathbf{k}, A = A(x, y, z), P = P(\xi, \eta, \zeta).$$

Since $\Delta(1/r_{AP}) = 0, A \notin V, P \in V$, we have

$$\Delta u(A) = \iiint_V \rho(P) \Delta(1/r_{AP}) dV = 0, \quad A \notin V.$$

Thus, the potential $u(A)$ of masses or charges distributed over volume V satisfies the Laplace equation at all points outside V . Away from the origin or, which is the same, from the domain V we have the approximate equality

$$u(A) \approx \frac{1}{r} \iiint_V \rho(P) dV = \frac{M}{r}, \quad (12)$$

where $M = \iiint_V \rho dV$ is the total mass. In other words, at infinity the potential of volume distributed masses (or charges) behaves like the potential of a mass point (or of a point charge) located at the origin such that its mass or charge is equal to the total mass (or to the total charge) distributed over volume V . In particular, $u(A) \rightarrow 0$ as $r \rightarrow \infty$. For partial derivatives of the potential of volume distributed masses we have the estimate

$$\left| \frac{\partial u}{\partial x} \right| < \frac{C}{r^2}, \quad \frac{\partial u}{\partial y} < \frac{C}{r^2}, \quad \left| \frac{\partial u}{\partial z} \right| < \frac{C}{r^2}, \quad (13)$$

where C is some constant.

2.2.3. Potential of a Homogenous Sphere

Assume that a sphere V of radius R with the center at the origin has constant density $\rho = \text{const}$. Passing to spherical coordinates r, φ, θ where $\xi = r \sin \theta \cos \varphi, \eta = r \sin \theta \sin \varphi, \zeta = r \cos \theta$, we obtain the potential of a homogenous sphere at a point r :

$$u(r) = \begin{cases} \frac{M}{r}, & \text{if } r > R, \\ \frac{M}{2R} \left\{ 3 - \left(\frac{r}{R} \right)^2 \right\}, & \text{if } r < R. \end{cases}$$

It is easy to see that $u(r)$ and its first-order derivative $u'(r)$ are continuous for all $r \geq 0$, but the second-order derivative $u''(r)$ becomes discontinuous at the point $r = R$.

At all exterior points the potential of a homogenous sphere is equal to the potential of the mass point of the same mass, placed at its centre, and satisfies the Laplace equation. At all interior points of the sphere the potential satisfies the Poisson equation $\Delta u = -4\pi\rho$.

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Biographical Sketches

Agoshkov Valery Ivanovich is a Doctor of Physical and Mathematical Sciences, professor of Institute of Numerical Mathematics of Russian Academy of Sciences (Moscow). He is the expert in the field of computational and applied mathematics, the theory of boundary problems for the partial differential equations and transport equation, the theory of the conjugate operators and their applications. He is also the author of more than 160 research works, including 9 monographs. His basic research works are devoted to:

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the development of methods of the optimal control theory and the theory of conjugate equations and their applications in the inverse problems of mathematical physics;

the development and justification of new iterative algorithms of the inverse problems solution;

the development of the theory of functional spaces used in the theory of boundary problems for the transport equation;

the determination of new qualitative properties of the conjugate equations solution.

Dubovski Pavel Borisovich is a Doctor of Physical and Mathematical Sciences, docent of Institute of Numerical Mathematics of Russian Academy of Sciences (Moscow). He is the expert in the field of differential and integral equations, the theory of Smolukhovsky equations, mathematical modeling. He is the author of more than 40 research works, including one monograph. His basic works are devoted to the development of the mathematical theory of coagulation and crushing kinetics, including the revelation of new kinetic models and transition to a hydrodynamic limit, the development of the theory of integral equations and nonlinear equations in partial derivatives, and the research of some problems of hydrodynamics.