NONLINEAR MODELS AND PLASTICITY

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Keywords: plastic solid, plastic deformation, stress space, loading surface, slip system, dislocation, plastic work, crack initiation.

Contents

UNESCO – EOLSS SAMPLE CHAPTERS

Summary

In mechanics, linear models have been well developed and received wide acceptance. They are based on linear elasticity, incompressibility of fluids and ideal gas properties. At the present stage of scientific and technical advances very exacting requirements are imposed upon materials and structures. The necessity to take into account their properties in nonlinear area of the mechanical behavior has given rise to the development of models both for theoretical description and prediction of nonlinear characteristics. Solutions involving the resulting complex mathematical expressions are made possible through the use of numerical solution programs on powerful computers. The nonlinear behavior in the mechanics of a deformable solid body is considered in the theories of plasticity, creep, and viscoelasticity. Although theory of plasticity is well developed, it must be noted that it is still far from being complete.

1. Plasticity

Strains, which do not disappear after removal of a load, are termed plastic. The plastic strain has in its basis a dislocation mechanism. The value of a plastic strain depends not only on a finite value of operating forces, but also on the order of their application, on the loading rate and on the duration for which the body is under the action of the load. However, in the development of models for plastic bodies it is desirable to separate these effects.

The typical stress-strain relation in the case of a body under simple tension load when the stress monotonically increases and exceeds the elasticity limit, is shown in Figure 1:

$$\sigma = f(\varepsilon), \qquad d\sigma > 0. \tag{1}$$

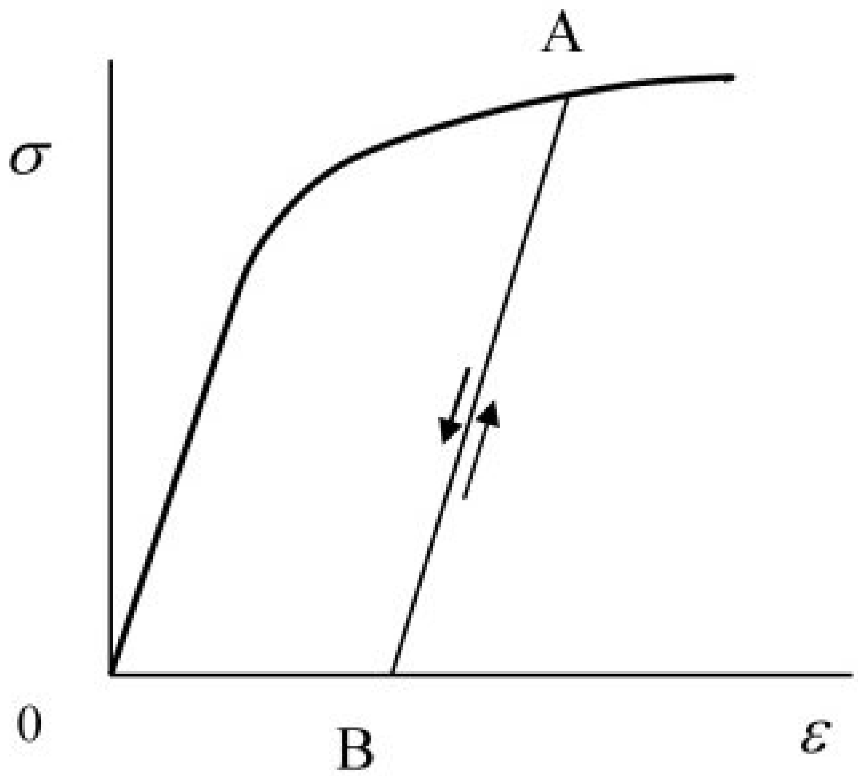

Figure 1: Typical stress-strain relation for an elastic-plastic body under tension.

While developing a plastic body model, it is assumed that this equation is valid at any loading program, in which the stress monotonically increases irrespective of the loading rate. For real materials the diagram of plastic deformation depends on the loading rate. However, for majority of constructional materials, this dependence is rather weak.

Upon unloading, the plastic deformation remains constant, whereas the elastic deformation disappears completely. The relationship between σ and ε is represented at unloading as a straight line AB (Fig. 1). The inclination of line AB is defined by the elasticity modulus. This is well supported by metals. The re-applied load follows the elasticity law until the stress at the point A is reached. The relationship between σ and ε is again depicted as line BA. Beyond the point A, a relation (1) will come into force again.

In soft steels, titanium alloys and certain types of other materials, a transition from elastic state to the plastic one occurs very suddenly and, therefore, the plastic part of diagram is almost horizontal. In the ideal case (Fig. 2), the stress will not exceed σ_y, which is called the yielding limit.

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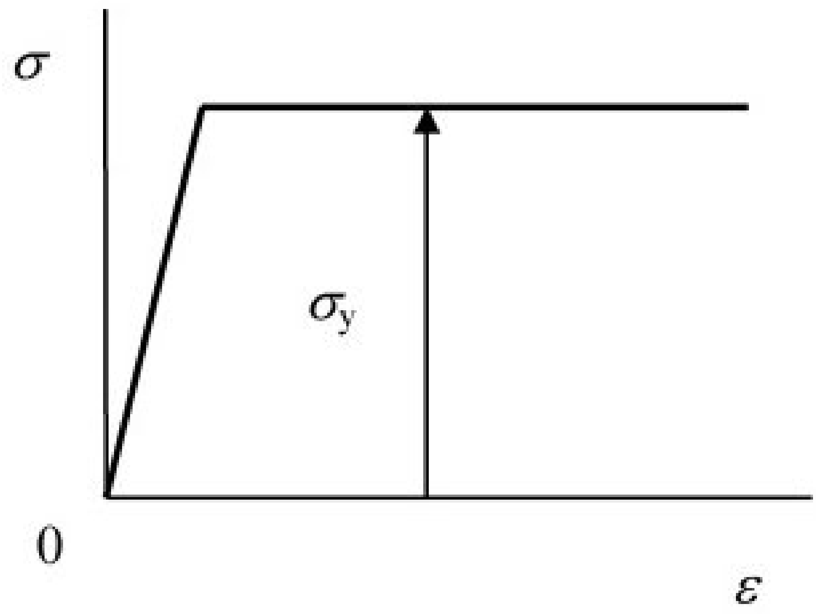

Figure 2: Characteristic of ideal plasticity.

2. Theory of Slipping

A polycrystalline body is an aggregate, which consists of a multitude of crystallites having various forms and orientation. In the theory of dislocation plastic deformation of each monocrystal is considered to occur in the planes, in directions of the closest packing molecules (planes and directions of slipping). For example, in a face-centered cubic lattice of crystal (Fig. 3), these are planes which intersect it on the face diagonals, and the directions are these diagonals. Each occurrence of a slipping plane and direction are together called the slipping system and is defined by a unit vector normal to the slipping plane $\boldsymbol{n}$ and by unit vector of slipping direction $\boldsymbol{t}$. Because in this case there are four planes and three directions of slipping, the face-centered lattice has 12 slipping systems.

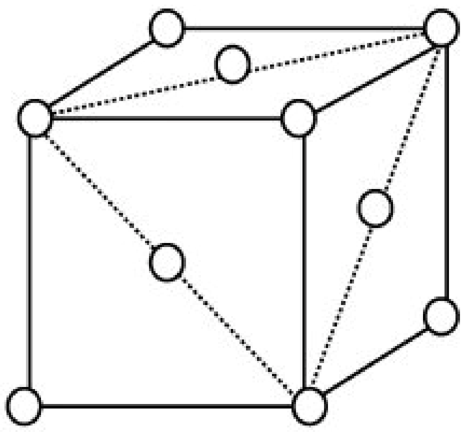

Figure 3: A face-centered cubic lattice of a crystal.

UNESCO – EOLSS SAMPLE CHAPTERS

Plastic deformation of shear in the slipping systems occurs by tangential stress according to the dislocation mechanism. As a first approximation, it is reasonable to describe the shear γ_{nt}^{p} in the slipping system $\boldsymbol{nt}$ by the component of shearing stress:

$$\gamma_{nt}^{p} = F(\tilde{\tau}_{nt}), \tag{2}$$

where, $\tilde{\tau}_{nt}$ is an extreme value of τ_{nt} for the deforming history in the given slipping system, exceeding some value τ_0 in magnitude at the initial time.

Let us take up a mobile coordinate system x_i' with the unit basis, two of which coincide with vectors $\boldsymbol{n}'$ and $\boldsymbol{t}'$ of some slipping system. In such coordinate system the shear stress component $\tau_{n't'}$ is defined on the basis of the tensor transformation law:

$$\tau_{n't'} = \sigma_{ij} n_i' t_i' \tag{3}$$

and, in view of (2), defines the shear $\gamma_{n't'}^{p}$ in the slipping system $\boldsymbol{n}', \boldsymbol{t}'$, and corresponding contribution to the plastic deformation of the whole aggregate in axes x_i :

$$\left(\varepsilon_{ij}^{p}\right)_0 = \frac{1}{2}\gamma_{n't'}^{p} n_i' t_i' \frac{N'}{N}, \tag{4}$$

where, N is a total number of grains, and N' is the number of grains oriented to the sliding system $\boldsymbol{n}', \boldsymbol{t}'$. This is taken into account, in the light of the fact that by virtue of tangential stress pairing law a shear simultaneously arises also in the pair slipping system of other crystallites, where the roles of $\boldsymbol{n}'$ and $\boldsymbol{t}'$ change places, the total plastic deformation of the aggregate can be represented by the formula:

$$\varepsilon_{ij}^{p} = \frac{1}{2}\sum_{n't'}\gamma_{n't'}^{p}\left(n_i' t_j' + n_j' t_i'\right)\frac{N'}{N} \tag{5}$$

Such theory, namely a theory of slipping, was proposed by Batdorf and Budiansky in 1947. The basic assumptions are as follows:

1. The stress states of every crystallite are identical and also coincide with those, which would exist at macro-homogeneity of a body.
2. Influence of the crystallite form and sizes on the aggregate deformation is neglected. In essence, an abstract monocrystal is considered instead of the crystallites.
3. Each monocrystal has only one slipping system oriented with equal probability.
4. The irreversible shear in the given slipping system $\boldsymbol{nt}$ depends only on the tangential component τ_{nt} of the stress vector, operating in a plane with normal $\boldsymbol{n}$ in the direction of vector $\boldsymbol{t}$.
5. The total irreversible deformation is the sum of the shear deformation contributions in all active slipping systems, where $\tau_{nt} = \tilde{\tau}_{nt}$.

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Each of these assumptions is vulnerable. But, if the weaknesses of the second, third and partly fourth has a quantitative character, the first and fifth assumptions are questionable in a qualitative sense. The acceptance of the first one at a random orientation of crystallites is tantamount to considering the displacement breaks on their boundaries, and the fifth one is the use of superposition principle in nonlinear effects, to which a plasticity is undoubtedly concerned.

Curiously, under subsequent developments in this direction, the attention is focused on perfecting items 1 – 4, but the fifth one remained basically unchanged. Information on the various "physical" theories of plasticity can be adequately found in the following review.

A phenomenological variant proposed by Leonov and Shvajko, where interplay between the slippings in various planes was taken into account, is an example of improvement of the preceding theory. Notice, that such a modification of the slipping theory can lose the ability to treat in terms of the dislocation mechanisms. So, the Malmejster theory is associated not with dislocation mechanisms but with the twining mechanism. The microstructural exposition of other similar theories is very challenging or almost unknown.

3. The Deformation Theory

Until the appearance of slipping theory, the description of plastic effects of polycrystalline bodies was made on the basis of only phenomenological concepts. By the the late 1920s, development of the theory of ideal plasticity was almost complete. However, in the problem of plasticity with hardening, which became by then very urgent in connection with the drastically rising manufacture of aluminum alloys, only the first steps was made. An active experimental checking of the theory of hardening proposed by Hencky and Nadai took place. The theory consists in the following statement :

$$\begin{aligned} s_{ij} &= 2G_s(T)e_{ij} \text{ if } T = \tilde{T} \\ ds_{ij} &= 2G\,de_{ij} \text{ if } T < \tilde{T} \end{aligned} \tag{6}$$

where, s_{ij}, and e_{ij} are the stress and the strain deviators respectively; G_s is the secant modulus on the diagram of pure shear, G is the modulus of elasticity in shear, T is the intensity of the tangential stresses:

$$T^2 = \frac{1}{2} s_{ij} s_{ij}\,, \tag{7}$$

$\tilde{T}$ is the maximum value of T for the loading history, exceeding some value T_0. It was supposed that the material behavior is elastic with reference to the volume deformation:

$$\sigma_{(1)} = 3K\varepsilon_{(1)}\,, \tag{8}$$

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where K is the modulus of the volume elasticity, and $\sigma_{(1)}\varepsilon_{(1)}$ are the first invariants of the stress tensor and of the strain tensor respectively. Sources of such theory and details of its experimental check can be found in the bibliography.

Obviously, this theory known as deformation theory is a special case of the elasticity theory, complemented by the effect of unloading (the second relationship in (6)). According to this theory in the phase space of a stress deviator (in the multi-measuring Cartesian coordinate system, where values of the deviator components s_{ij} are plotted along the axes) for arbitrary point of body at each instant of loading, there is a limiting surface as the hypersphere $T = \tilde{T} =$ constant (at the initial moment $T = T_0$). If the vector $\mathbf{s}$, having components of the deviator s_{ij}, moves inside this limiting surface as a result of changing stress state, $T < \tilde{T}$,and, by virtue of second expression of (6), there is only elastic change of strains. In such passive process, the limiting surface does not vary. The plastic strains arise and vary, if this vector has punctured the initial limiting surface $T = T_0$ and changing it will be the longer than the previous values. Obviously, in such active process a limiting surface changes by itself, and its instantaneous radius, is the length of that vector. Thus, this vector can not puncture the instantaneous limiting surface. Attempts at such puncture tend to dilate the sphere and are accompanied by changing plastic deformation. Such limiting surface is referred to as a loading surface, and sometimes, as a yielding surface. Let the pointing out vector be named as the stress vector. In the following, we shall also extend this name to a vector with components of the stress tensor $\boldsymbol{\sigma}$ and not just of its deviator $\mathbf{s}$. The path traced by the stress vector tip is referred to as a trajectory or the loading path.

Henceforth, the vector representations of other bivalency tensors will be used, namely, the deformation tensor ε_{ij}, the deformation tensor $\varepsilon_{ij}^{\mathrm{p}}$, etc.

The deformation theory (6) has found a wide acceptance in the design practice due to its simplicity and satisfactory ability to predict conditions of external loading and individual properties of a material, as it ensures a small deviation of the loading path from radial one at each body point. The last is provided by the known theorem about simple loading of Iliouchine. The loading at each point of a body will be simple (radial), if the external loading is carried out by a rising surface and the mass forces, proportionally to one parameter. In doing so, a material is incompressible and the function $G_{\mathrm{s}}(T)$ is an ascending power function:

$$\varepsilon_{(1)} = 0, G_{\mathrm{s}}(T) = AT^{n} \tag{9}$$

The proof of this theorem is elemental. It is necessary to verify that, at the indicated conditions, it is possible to satisfy equilibrium equations, static boundary conditions, compatibility condition, and ratios (6) and (8) by taking $\sigma_{ij} = \lambda\sigma_{ij}^{0}$ where σ_{ij}^{0} is a fixed stress tensor. It is obvious, that a small deviation of properties, as specified in (9), at proportional external loading will result in a small deviation from the radial internal

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loading.

However, at a drastic deviation from the radial loading, the predictions by the deformation theory can be erroneous. So, at the loadings, close to so called neutral one (loading along the limiting surface), the serious defect of this theory comes to light. On two arbitrarily close intercepts of trajectories going from a given point *a* on the limiting surface, constructed for this point of the previous trajectory, the deformation increments inside and outside (Fig. 4) differ by a finite quantity. Indeed, as the external trajectory corresponds to active process, so

$$ds_{ij} = 2G_s(T)de_{ij} + 2G_s'(T)e_{ij}dT \,, \tag{10}$$

and the internal trajectory conforms to the passive one when

$$ds_{ij} = 2Gde_{ij} \,. \tag{11}$$

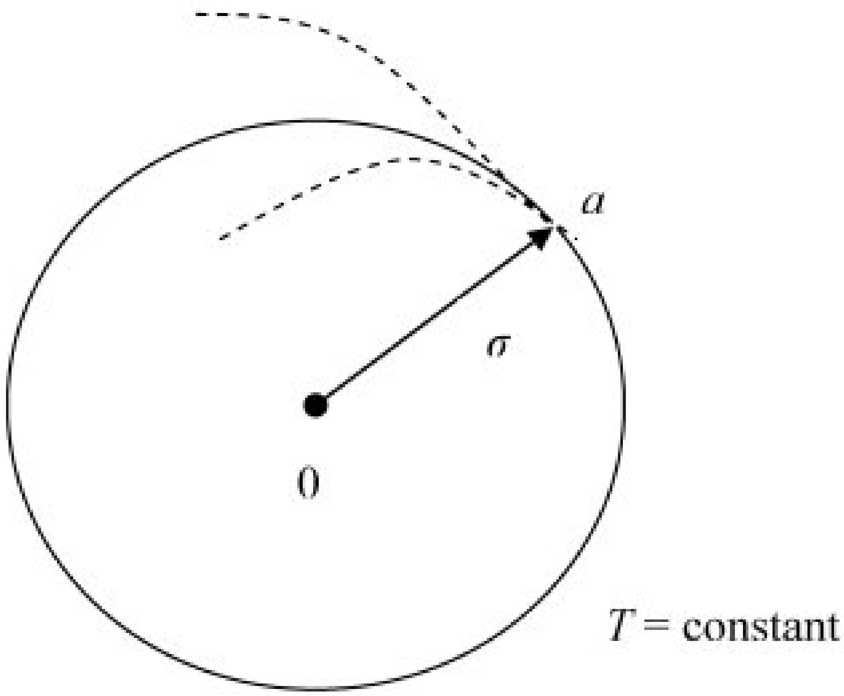

Figure 4: A schematic sketch of a limiting surface and the loading trajectories.

At the spanning of these trajectories together, that is at $T \to T_a$ $(dT \to 0)$ the ratio of elementary deformation increments is defined by the formula:

$$\frac{\left(de_{ij}\right)_{\text{act}}}{\left(de_{ij}\right)_{\text{pass}}} = \frac{G}{G_{\text{s}}\left(T\right)} \,, \tag{12}$$

and, as the secant modulus G_{s} in plastic area is not equal to the elastic modulus G, on the final intercepts of arbitrarily close trajectories the final difference of deformations will be accumulated.

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Biographical Sketch

Berezin Alexander V., born September 16, 1946, Russia. He graduated from Moscow State University, Department of Mathematics and Mechanics in 1969. He received the Degree of Candidate of Sciences in 1973 and the Degree of Doctor of Sciences in 1986.

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