DIGITAL CONTROL SYSTEMS

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Summary

Digital control methods have seen practical development over the last two decades. Up-

to-date digital controllers have now replaced most conventional analog types. This is due to the fact that controlling a system or a plant using a computer offers great advantages over conventional techniques. These include greater controller flexibility, simpler data processing, superior sensitivity, fewer drift effects, less internal noise, and greater reliability; the units are also cheaper and smaller in size. The major disadvantage in their use is the degree of error introduced during quantization.

From the theoretical and design points of view, digital and conventional methods essentially involve the same degree of difficulty. From the applications point of view, digital controllers are usually easier to apply. For this reason they have been successfully applied to almost all categories of control systems, ranging from position control systems to industrial processes, reactors, robots, weapon systems, space applications, and others.

1. The Basic Structure of Digital Control Systems

The basic structure of a typical digital control system, or a computer control system or discrete-time control system, is shown in Figure 1. The system (plant or process) under control is a continuous-time system (for example, a motor, electrical power plant, or robot). The “heart” of the controller is the digital computer. The A/D converter converts a continuous-time signal into a discrete-time signal at times specified by a clock. The D/A converter, by contrast, converts the discrete-time signal output of the computer to a continuous-time signal to be fed to the plant. The D/A converter normally includes a hold circuit (for more on A/D and D/A converters see *Discrete-Time Equivalents to Continuous Time Systems* (Subsections 2.2 and 2.3). The quantizer Q converts a discrete-time signal to binary digits.

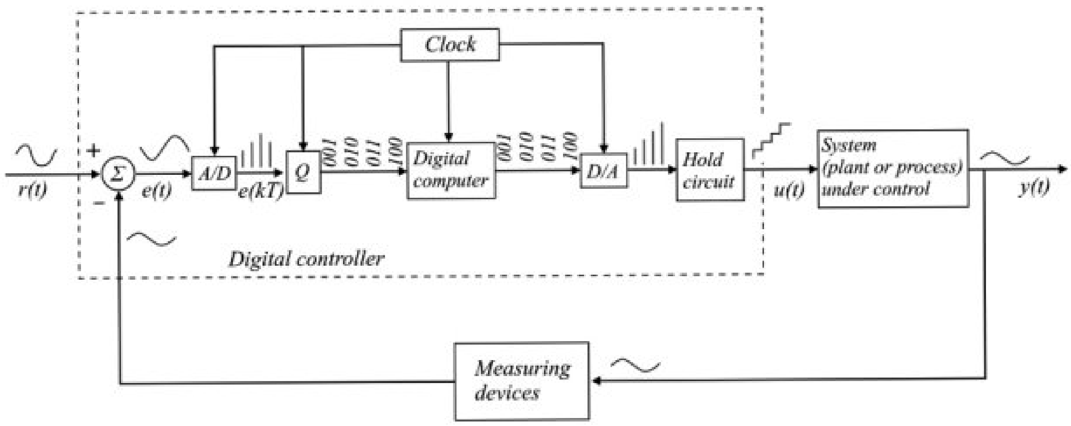

Figure 1. Simplified block diagram of a typical closed-loop digital control or computer-controlled system

2. Discrete-Time Systems

2.1. Introduction

The term discrete-time system covers systems that operate directly with discrete-time signals. In this case the input, as well as the output, of the system are both discrete-time signals (Figure 2). A well-known discrete-time system is the digital computer, wherein the signals *u(k)* and *y(k)* are number sequences. These types of system are described by difference equations, as opposed to continuous-time systems that are described by differential equations.

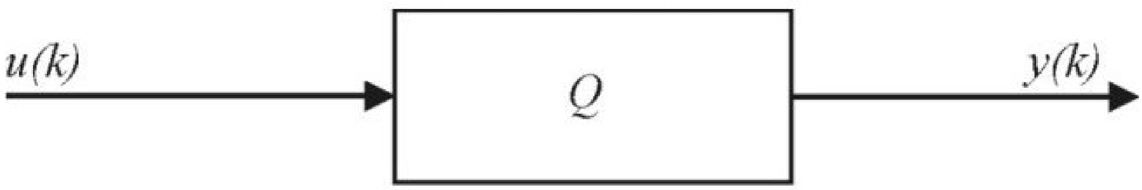

Figure 2. Block diagram of a discrete-time system

A linear, time-invariant, discrete-time system is described by a difference equation of the general form

$$\begin{aligned} y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) \\ = b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-m) \end{aligned} \tag{1}$$

From this, one may readily derive the special cases

$$y(k) + a_1 y(k-1) = b_0 u(k) + b_1 u(k-1) \tag{2a}$$

which is a first order difference equation, or of the form

$$\begin{aligned} y(k) + a_1 y(k-1) + a_2 y(k-2) \\ = b_0 u(k) + b_1 u(k-1) + b_2 u(k-2) \end{aligned} \tag{2b}$$

which is a second-order difference equation, and so on.

There are other popular ways to describe discrete-time systems; these include the transfer function, the impulse response or weight function, and the state-space equations. The *transfer function* of a discrete-time system is denoted by *H(z)* and is defined as the ratio of the Z-transform of the output *y(k)* divided by the Z-transform of the input u(k), under the condition that *u(k) = y(k) = 0,* for all negative values of k. That is

$$H(z) = \frac{Z[y(k)]}{Z[u(k)]} = \frac{Y(z)}{U(z)} \tag{3}$$

where u(k) = y(k) = 0 for k<0.

The *impulse response* (or *weight function*) of a system is denoted by *h(k)*. It is defined as the output of a system when its input is the unit impulse sequence $\delta(k)$, under the constraint that the initial conditions $y(-1), y(-2), \ldots, y(-n)$ of the system are zero.

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The transfer function *H(z)* and the weight function *h(k)* are related by

$$H(z) = Z[h(k)] \tag{4}$$

where $Z[f(k)]$ indicates the Z-transform of $f(k)$.

State-space equations, or simply *state equations*, are a set of first-order difference equations describing high-order systems, and have the form

$$\mathbf{x}(\mathrm{k}+1) = \mathbf{A}\mathbf{x}(\mathrm{k}) + \mathbf{B}\mathbf{u}(\mathrm{k}) \tag{5a}$$

$$\mathbf{y}(\mathrm{k}) = \mathbf{C}\mathbf{x}(\mathrm{k}) + \mathbf{D}\mathbf{u}(\mathrm{k}) \tag{5b}$$

where $\mathbf{u}(k) \in R^m$, $\mathbf{x}(k) \in R^n$, and $\mathbf{y}(k) \in R^p$ are the input, state, and output vectors, respectively and A, B, C, and D are constant matrices of appropriate dimensions.

2.2. Analysis of Linear Time-Invariant Discrete-Time Systems

The analysis problem involving linear, time-invariant, discrete-time systems that is considered here is as follows: given a mathematical description of a system, together with its initial conditions, find its output $y(k)$ for any given input $u(k)$.

When the system is described by a difference equation, the difference equation is solved to yield $y(k)$, either directly in the time domain or using the Z-transform.

When the system is described by its transfer function $H(z)$, in which case $Y(z) = H(z)U(z)$, then

$$y(k) = Z^{-1}[Y(z)] = Z^{-1}[H(z)U(z)] \tag{6}$$

When the system is described by its impulse response $h(k)$, then

$$y(k) = \sum_{i=0}^{\infty} u(i)h(k-i) = \sum_{i=0}^{\infty} u(k-i)h(i) \tag{7}$$

When the system is described in state-space, then the solution of Eq. (5) is as follows:

$$\mathbf{x}(k) = \mathbf{A}^k\mathbf{x}(0) + \sum_{i=0}^{k-1} \mathbf{A}^{k-i-1}\mathbf{B}u(i) \tag{8a}$$

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$$\mathbf{y}(k) = \mathbf{C}\mathbf{A}^k\mathbf{x}(0) + \mathbf{C}\sum_{i=0}^{k-1}\mathbf{A}^{k-i-1}\mathbf{B}u(i) + \mathbf{D}u(k) \qquad (8b)$$

3. Sampled-Data Systems

3.1. Introduction

The term *sampled-data systems* covers normal analog (continuous-time) systems with the distinctive characteristic that the input u(t) and the output y(t) are piecewise constant signals: in other words, they are constant over each interval between two consecutive sampling points. This means that sampled-data systems may be described and subsequently studied similarly to discrete-time systems. This fact is of particular importance because it unifies the study of hybrid systems that consist of continuous-time and discrete-time subsystems using a common mathematical tool, namely the difference equation. For this reason, and for reasons of simplicity, sampled-data systems are usually addressed in the literature as discrete-time systems. It is noted that sampled-data systems are also called *discretized systems*.

3.2. Description and Analysis of Sampled-Data Systems

The article *Discrete-Time, Sampled Data, Digital Control Systems, and Quantization Effects* that follows addresses the derivation of discrete-time equivalents to continuous-time systems. The discrete-time equivalents are difference equations, and constitute the time-domain description of the sampled-data systems. Using the Z-transform, one may readily derive the transfer function of the sampled-data systems.

On the basis of the discrete-time equivalent descriptions one may proceed to the analysis problem by directly extending the results of Section 3.1 to the present case. To exemplify this, consider the continuous-time, multi-input, multi-output, open-loop system described in state-space by the first order differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{m}(t) \qquad (10a)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{m}(t) \qquad (10b)$$

Then, the sampled-data description of Eq. (10) is the following

$$\mathbf{x}[(k+1)T] = \mathbf{A}(T)\mathbf{x}(kT) + \mathbf{B}(T)\mathbf{u}(kT) \qquad (11a)$$

$$\mathbf{y}(kT) = \mathbf{C}\mathbf{x}(kT) + \mathbf{D}\mathbf{u}(kT) \qquad (11b)$$

where T is the sampling period, and where

$$\mathrm{A(T)} = e^{\mathbf{F}T} \qquad (12a)$$

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$$\mathbf{B}(T)=\int_0^T e^{\mathbf{F}(T-\xi)}\mathbf{G}d\xi=\int_0^T e^{\mathbf{F}\lambda}\mathbf{G}d\lambda,\lambda=\mathrm{T}-\xi \tag{12b}$$

3.2.1. Example 1

Consider the continuous-time system

$$\dot{\mathbf{x}}(t)=\mathbf{F}\mathbf{x}(t)+\mathbf{g}\mathbf{m}(t)$$

$$y(t)=\mathbf{c}^T\mathbf{x}(t)$$

Where

$$\mathbf{F}=\begin{bmatrix}-1 & 0\\ 1 & 0\end{bmatrix},\quad \mathbf{g}=\begin{bmatrix}2\\ 1\end{bmatrix},\quad \mathbf{c}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

Find the equivalent discrete-time (sampled-data) system: in other words, find the matrix A(T) and the vector b(T).

Solution: We have

$$s\mathbf{I}-\mathbf{F}=\begin{bmatrix}s+1 & 0\\ -1 & s\end{bmatrix},\quad (s\mathbf{I}-\mathbf{F})^{-1}=\begin{bmatrix}\dfrac{1}{s+1} & 0\\ \dfrac{1}{s(s+1)} & \dfrac{1}{s}\end{bmatrix}$$

and

$$L^{-1}\left[(s\mathbf{I}-\mathbf{F})^{-1}\right]=\begin{bmatrix}e^{-t} & 0\\ 1-e^{-t} & 1\end{bmatrix}$$

Therefore

$$\mathbf{A}(T)=\left[L^{-1}\left[s\mathbf{I}-\mathbf{F}\right]^{-1}\right]_{t=T}=e^{\mathbf{F}T}=\begin{bmatrix}e^{-T} & 0\\ 1-e^{-T} & 1\end{bmatrix}$$

Moreover,

$$\mathbf{b}(T)=\int_0^T e^{\mathbf{F}\lambda}\mathbf{g}d\lambda$$

$$=\int_0^T\begin{bmatrix}e^{-\lambda} & 0\\ 1-e^{-\lambda} & 1\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix}d\lambda=\int_0^T\begin{bmatrix}2e^{-\lambda}\\ 3-2e^{-\lambda}\end{bmatrix}d\lambda=\begin{bmatrix}2(1-e^{-T})\\ 3T-2(1-e^{-T})\end{bmatrix}$$

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3.2.2. Example 2

Consider a harmonic oscillator with transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\omega^2}{s^2 + \omega^2}$$

A state-space description of the oscillator is of the form $\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x} + \mathbf{g}\mathbf{m}$, $y(t) = \mathbf{c}^T \mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \omega^{-1} y^{(1)} \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \ \mathbf{g} = \begin{bmatrix} 0 \\ \omega \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Find A(T) and b(T) of the discretized model.

Solution: We have

$$(s\mathbf{I} - \mathbf{F})^{-1} = \begin{bmatrix} \dfrac{s}{s^2 + \omega^2} & \dfrac{\omega}{s^2 + \omega^2} \\ \dfrac{-\omega}{s^2 + \omega^2} & \dfrac{s}{s^2 + \omega^2} \end{bmatrix}$$

and

$$e^{\mathbf{F}T} = \begin{bmatrix} \cos\omega T & \sin\omega T \\ -\sin\omega T & \cos\omega T \end{bmatrix}$$

Therefore

$$\mathbf{A}(T) = \left[L^{-1} \left[s\mathbf{I} - \mathbf{F} \right]^{-1} \right]_{t=T} = \begin{bmatrix} \cos\omega T & \sin\omega T \\ -\sin\omega T & \cos\omega T \end{bmatrix}$$

$$\mathbf{b}(T) = \left[\int_0^T e^{\mathbf{F}T} \mathbf{g} d\lambda \right] = \begin{bmatrix} 1 - \cos\omega T \\ \sin\omega T \end{bmatrix}$$

3.3. Analysis of Sampled-Data Systems

In order to solve Eq. (11), we take advantage of the results of Section 3.2 since they

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differ only in the constant T. Solving Eq. (11) for $\mathbf{x}(kT)$ and $\mathbf{y}(kT)$ yields

$$\mathbf{x}(kT) = \Phi[(k-k_0)T]\mathbf{x}(k_0T) + \sum_{i=k_0}^{k-1} \Phi[(k-i-1)T]\mathbf{B}(T)\mathbf{u}(iT) \tag{13a}$$

$$\mathbf{y}(kT) = \mathbf{C}\Phi[(k-k_0)T]\mathbf{x}(k_0T) + \mathbf{C}\sum_{i=k_0}^{k-1} \Phi[(k-i-1)T]\mathbf{B}(T)\mathbf{u}(iT) + \mathbf{D}\mathbf{u}(kT) \tag{13b}$$

where $\Phi[(k-k_0)T]$ is the transition matrix, given by the relation $\Phi[(k-k_0)T] = [\mathbf{A}(T)]^{k-k_0}$.

Clearly, if we set T = 1 in Eq. (13), then we obtain the corresponding discrete-time results given in Eq. (8).

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Biographical Sketch

P.N. Paraskevopoulos was born in Doxa Arkadias, Greece, in 1941. He received his Bachelor’s (1964) and Master’s (1965) Degrees from Illinois Institute of Technology, and his Ph.D. from the University of Patras, Greece (1976). He has been Professor of Control in the Democritus University of Thrace (1977–1985) and in the National Technology University of Athens (1985 to date). He has published over 130 journal papers and 70 conference papers in the field of control engineering. He has written ten books on control in Greek and two in English (*Digital Control Systems*, Prentice Hall, London, 1996; *Modern Control Engineering*, Marcel Dekker, New York, 2001). He is the founder and director of the Control Systems Laboratory, which is considered to be among the best of its kind in Europe. His current research and development interests are mainly in the following areas: system identification; system theory; controller design for linear and nonlinear multivariable systems; computer-controlled systems; optimal and algebraic control; adaptive and robust control; control of discrete-event systems; industrial applications, particularly in the processing industry (for example, refining, paper, cement, plastic, mill, aluminum, and food industries).