SYSTEM CHARACTERISTICS: STABILITY, CONTROLLABILITY, OBSERVABILITY

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Contents

Summary

In this article, linear, continuous-time, finite-dimensional control systems with constant coefficients are considered. The article is divided into three main parts. The first part contains fundamental definitions of stability and necessary and sufficient conditions for stability. In the second part controllability of dynamical control system is defined and, using the controllability matrix, necessary and sufficient conditions for controllability are presented. Additionally, the important case of controllability with constrained controls is also discussed. The third part is devoted to a study of observability. In this part necessary and sufficient observability conditions are formulated using the observability matrix. In conclusion, several remarks concerning special cases of stability, controllability, and observability of linear control systems are given. It should be noted that all the results are given without proofs but with suitable literature references.

1. Introduction

Stability, controllability, and observability are among the fundamental concepts in modern mathematical control theory. They are qualitative properties of control systems and are of particular importance in control theory. Systematic study of controllability

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and observability was started at the beginning of the 1960s, when the theory of controllability and observability based on a description in the form of state space for both time-invariant and time-varying linear control systems was worked out. The concept of stability is extremely important, because almost every workable control system is designed to be stable. If a control system is not stable, it is usually of no use in practice. Many dynamical systems are such that the control does not affect the complete state of the dynamical system but only a part of it. On the other hand, in real industrial processes it is very often possible to observe only a certain part of the complete state of the dynamical system. Therefore, it is very important to determine whether or not control and observation of the complete state of the dynamical system are possible. Roughly speaking, controllability generally means, that it is possible to steer a dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. On the other hand, observability means that it is possible to recover the initial state of the dynamical system from knowledge of the input and output.

Stability, controllability, and observability play an essential role in the development of modern mathematical control theory. There are important relationships between stability, controllability, and observability of linear control systems. Controllability and observability are also strongly connected with the theory of minimal realization of linear time-invariant control systems. It should be pointed out that a formal duality exists between the concepts of controllability and observability.

In the literature there are many different definitions of stability, controllability, and observability, which depend on the type of dynamical control system. The main purpose of this article is to present a compact review of the existing stability, controllability, and observability results mainly for linear continuous-time and time-invariant control systems. It should be noted that for linear control systems, stability, controllability, and observability conditions have pure algebraic forms and are fairly easily computable. These conditions require verification, location of the roots of a characteristic polynomial, and of the rank conditions for suitable defined constant controllability and observability matrices.

The article is organized as follows. Section 2 contains systems descriptions and fundamental results concerning the solution of the most popular linear continuous-time control models with constant coefficients. Section 3 is devoted to a study of different kinds of stability. Section 4 presents fundamental definitions of controllability and necessary and sufficient conditions for different kinds of controllability. Section 5 contains fundamental definitions of observability, and necessary and sufficient conditions for observability. Finally, in Section 6 concluding remarks and comments concerning possible extensions are presented. For reasons of space, it is impossible to give a full survey of the subject. In consequence, only selected fundamental results without proofs are presented.

2. Mathematical Model

In the theory of linear time-invariant dynamical control systems the most popular and the most frequently used mathematical model is given by the following differential state equation and algebraic output equations

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$x'(t) = Ax(t) + Bu(t)$ (1)

$y(t) = Cx(t)$ (2)

where $x(t)\in R^n$ is a state vector, $u(t)\in R^m$ is an input vector, $y(t)\in R^p$ is an output vector, and A, B, and C are real matrices of appropriate dimensions.

It is well known that for a given initial state $x(0)\in R^n$ and control $u(t)\in R^m$, $t\geq 0$, there exist a unique solution $x(t;x(0),u)\in R^n$ of the state equation (1) of the following form

$$x(t;x(0),u) = \exp(At)x(0) + \int_0^t \exp(A(t-s))Bu(s)ds$$

Let P be an $n \times n$ constant nonsingular transformation matrix and let us define the equivalence transformation $z(t)=Px(t)$. Then the state equation (1) and output equation (2) becomes

$z'(t) = Jz(t) + Gu(t)$ (3)

$y(t) = Hz(t)$ (4)

where $J=PAP^{-1}$, $G=PB$ and $H=CP^{-1}$.

Dynamical systems (1), (2), and (3), (4) are said to be equivalent and many of their properties are invariant under the equivalence transformations. Among different equivalence transformations special attention should be paid on the transformation, which leads to the so-called Jordan canonical form of the dynamical system. If the matrix J is in the Jordan canonical form, then Eqs. (3) and (4) are said to be in a Jordan canonical form. It should be stressed, that that every dynamical system (1), (2) has an equivalent Jordan canonical form.

3. Stability

In order to introduce the stability definitions we need the concept of equilibrium state.

Definition 1: A state x^e of a dynamical system (1) is said to be an equilibrium state if and only if $x^e=x(t;x^e,0)$ for all $t\geq 0$.

We see from this definition that if a trajectory reaches an equilibrium state and if no input is applied, the trajectory will stay at the equilibrium state forever. Clearly, for linear dynamical systems the zero state is always an equilibrium state.

Definition 2: An equilibrium state x^e is said to be stable if and only if for any positive ε, there exists a positive number $\delta(\varepsilon)$ such that $\left\| x(0) - x^e \right\| \leq \delta$ implies that

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$\left\| x(t;x(0),0) - x^e \right\| \leq \varepsilon$ for all t≥0.

Roughly speaking, an equilibrium state x^e is stable if the response due to any initial state that is sufficiently near to x^e will not move far away from x^e. If the response will, in addition, go back to x^e, then x^e is said to be asymptotically stable.

Definition 3: An equilibrium state x^e is said to be asymptotically stable if there is some $\gamma \geq 0$, and for every positive ε there corresponds a positive $T(\varepsilon,\gamma)$, independent of x(0), such that $\left\| x(0) - x^e \right\| \leq \delta$ implies that $\left\| x(t;x(0),0) - x^e \right\| \leq \varepsilon$ for all t≥T.

In other words an equilibrium state x^e is said to be asymptotically stable if it is stable in the sense of Lyapunov and if every motion starting sufficiently near to x^e converges to x^e as $t \to \infty$.

Let $s_i = \mathrm{Re}(s_i) + j\mathrm{Im}(s_i)$, i=1,2,3,...,r, r≤n denote the distinct eigenvalues of the matrix A and let "Re" and "Im" stand for the real part and the imaginary part of the eigenvalue s_i, respectively.

Theorem 1: Every equilibrium state of the dynamical system (1) is stable if and only if all the eigenvalues of A have nonpositive (negative or zero) real parts, i.e., $\mathrm{Re}(s_i) \leq 0$ for i=1,2,3,...,r and those with zero real parts are simple zeros of the minimal polynomial of A.

Theorem 2: The zero state of the dynamical system (1) is asymptotically stable if and only if all the eigenvalues of A have negative real parts, i.e., $\mathrm{Re}(s_i) < 0$ for i=1,2,3,...,r.

From the above theorems it directly follows that the stability and asymptotic stability of a dynamical system depend only on the matrix A and are independent of the matrices B and C. Suppose that the dynamical system (1) is stable or asymptotically stable, then the dynamical system remains stable or asymptotically stable after arbitrary equivalence transformation. This is natural and intuitively clear because an equivalence transformation changes only the basis of the state space. Therefore, we have the following corollary.

Corollary 1: Stability and asymptotic stability are both invariant under any equivalence transformation.

4. Controllability

4.1. Fundamental Results

Let us recall the most popular and frequently used fundamental definition of controllability for linear control systems with constant coefficients.

Definition 4: Dynamical system (1) is said to be controllable if for every initial condition x(0) and every vector $x^1 \in R^n$, there exist a finite time t_1 and control $u(t) \in R^m$,

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$t \in [0,t_1]$, such that $x(t_1;x(0),u)=x^1$.

This definition requires only that any initial state x(0) can be steered to any final state x^1. The trajectory of the system is not specified. Furthermore, there are no constraints imposed on the control. In order to formulate easily computable algebraic controllability criteria let us introduce the so-called controllability matrix W defined as follows:

$$W = [B, AB, A^2B, ..., A^{n-1}B].$$

Controllability matrix W is an $n \times nm$-dimensional constant matrix and depends only on system parameters.

Theorem 3: Dynamical system (1) is controllable if and only if

$$\text{rank } W = n$$

Corollary 2: Dynamical system (1) is controllable if and only if the $n \times n$-dimensional symmetric matrix WW^T is nonsingular.

Since the controllability matrix W does not depend on time t_1, then from Theorem 3 and Corollary 2 it directly follows that in fact the controllability of a dynamical system does not depend on the length of control interval. Let us observe that in many cases, in order to check controllability it is not necessary to calculate the controllability matrix W but only a matrix with a smaller number of columns. It depends on the rank of the matrix B and the degree of the minimal polynomial of the matrix A, where the minimal polynomial is the polynomial of the lowest degree that annihilates matrix A. This is based on the following corollary.

Corollary 3: Let rank B = r, and q is the degree of the minimal polynomial of the matrix A. Then dynamical system (1) is controllable if and only if

$$\text{rank}[B, AB, A^2B, ..., A^{n-k}B] = n$$

where the integer $k \le \min(n-r, q-1)$.

In a case where the eigenvalues of the matrix A, s_i, i=1,2,3,...,n are known, we can check controllability using the following corollary.

Corollary 4: Dynamical system (1) is controllable if and only if

$$\text{rank}[s_iI-A|B] = n \qquad \text{for all } s_i,\ i=1,2,3,...,n$$

Suppose that the dynamical system (1) is controllable, then the dynamical system remains controllable after the equivalence transformation. This is natural and intuitively clear because an equivalence transformation changes only the basis of the state space. Therefore, we have the following corollary.

Corollary 5: Controllability is invariant under any equivalence transformation.

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Since controllability of a dynamical system is preserved under any equivalence transformation, then it is possible to obtain a simpler controllability criterion by transforming the differential state equation (1) into a special form (3). If we transform dynamical system (1) into Jordan canonical form, then controllability can be determined very easily, almost by inspection.

4.2. Stabilizability

It is well known that the controllability concept for dynamical system (1) is strongly related to its stabilizability by the linear static state feedback of the following form

$$u(t) = Kx(t) + v(t) \tag{5}$$

where $v(t) \in R^m$ is a new control, K is $m \times n$-dimensional constant state feedback matrix.

Introducing the linear static state feedback given by equality (5) we directly obtain the linear differential state equation for the feedback linear dynamical system of the following form

$$x'(t) = (A + BK)x(t) + Bv(t) \tag{6}$$

which is characterized by the pair of constant matrices (A + BK, B).

An interesting result is the equivalence between controllability of the dynamical systems (1) and (6), explained in the following corollary.

Corollary 6: Dynamical system (1) is controllable if and only if for an arbitrary matrix K the dynamical system (6) is controllable.

From Corollary 6 it follows that under the controllability assumption we can arbitrarily form the spectrum of the dynamical system (1) by the introduction of suitable defined linear static state feedback (5). Hence, we have the following result.

Theorem 4: The pair of matrices (A,B) represents the controllable dynamical system (1) if and only if for each set Λ consisting of n complex numbers and symmetric with respect to real axis, there exists constant state feedback matrix K such that the spectrum of the matrix (A+BK) is equal to the set Λ.

Practically, in the design of the dynamical system, it is sometimes only necessary to change unstable eigenvalues (that is, the eigenvalues with nonnegative real parts) into stable eigenvalues (i.e., the eigenvalues with negative real parts). This is called stabilization of the dynamical system (1). Therefore, we have the following formal definition of stabilizability.

Definition 5: The dynamical system (1) is said to be stabilizable if there exists a constant static state feedback matrix K such that the spectrum of the matrix (A+BK) entirely lies in the left-hand side of the complex plane.

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Let $Re(s_j) \geq 0$, for $j=1,2,3,...,q \leq n$; in other words, s_j are unstable eigenvalues of the dynamical system (1). An immediate relation between controllability and stabilizability of the dynamical system (1) gives the following theorem.

Theorem 5: The dynamical system (1) is stabilizable if and only if all its unstable modes are controllable, that is,

$$\operatorname{rank}[s_j I - A | B] = n \qquad \text{for } j=1,2,3,...,q$$

Comparing Theorem 4 and Corollary 4 we see, that controllability of the dynamical system (1) always implies its stabilizability, but the converse statement is not always true. Therefore, the stabilizability concept is essentially weaker than the controllability one.

4.3. Output Controllability

Similar to the state controllability of a dynamical control system, it is possible to define the so-called output controllability for the output vector y(t) of a dynamical system. Although these two concepts are quite similar, it should be mentioned that the state controllability is a property of the differential state equation (1), whereas the output controllability is a property both of the state equation (1) and algebraic output equation (2).

Definition 6: Dynamical system (1), (2) is said to be output controllable if for every y(0) and every vector $y^1 \in R^p$, there exist a finite time t_1 and control $u^1(t) \in R^m$, that transfers the output from y(0) to $y^1 = y(t_1)$.

Theorem 6: Dynamical system (1), (2) is output controllable if and only if

$$\operatorname{rank}[CB, CAB, CA^2B, ..., CA^{n-1}B] = p$$

It should be pointed out, that the state controllability is defined only for the linear differential state equation (1), whereas the output controllability is defined for the input-output description, that is, it depends also on the linear algebraic output equation (2). Therefore, these two concepts are not necessarily related.

If the control system is output controllable, its output can be transferred to any desired vector at certain instant of time. A related problem is whether it is possible to steer the output following a preassigned curve over any interval of time. A control system whose output can be steered along the arbitrary given curve over any interval of time is said to be output function controllable or functional reproducible.

4.4. Controllability with Constrained Controls

In practice admissible controls are required to satisfy additional constraints. Let $U \subset R^m$ be an arbitrary set and let the symbol M(U) denotes the set of admissible controls, i.e., the set of controls $u(t) \in U$ for $t \in [0, \infty)$.

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Definition 7: The dynamical system (1) is said to be U-controllable to zero if for any initial state $x(0)\in R^n$, there exist a finite time $t_1<\infty$ and an admissible control $u(t)\in M(U)$, $t\in[0,t_1]$, such that $x(t_1; x(0),u) = x^1$.

Definition 8: The dynamical system (1) is said to be U-controllable from zero if for any final state $x^1\in R^n$, there exist a finite time $t_1<\infty$ and an admissible control $u(t)\in M(U)$, $t\in[0,t_1]$, such that $x(t_1;0,u) = x^1$.

Definition 9: The dynamical system (1) is said to be U-controllable if for any initial state $x(0)\in R^n$, and any final state $x^1\in R^n$, there exist a finite time $t_1<\infty$ and an admissible control $u(t)\in M(U)$, $t\in[0,t_1]$, such that $x(t_1; x(0),u) = x^1$.

Generally, for arbitrary set U it is rather difficult to give easily computable criteria for constrained controllability. However, for certain special cases of the set U it is possible to formulate and prove algebraic constrained controllability conditions.

Theorem 7: The dynamical system (1) is U-controllable to zero if and only if all the following conditions are satisfied simultaneously:

1. There exists $w\in U$ such that $Bw=0$.
2. The convex hull CH(U) of the set U has nonempty interior in the space R^m.
3. $\mathrm{Rank}[B,AB,A^2B,...,A^{n-1}B] = n$.
4. There is no real eigenvector $v\in R^n$ of the matrix A^{tr} satisfying $v^{tr}Bw\leq 0$ for all $w\in U$.
5. No eigenvalue of the matrix A has a positive real part.

For the single input system, that is, m=1, Theorem 7 reduces to the following corollary:

Corollary 7: Suppose that m=1 and U=[0,1]. Then the dynamical system (1) is U-controllable to zero if and only if it is controllable without any constraints; that is, $\mathrm{rank}[B,AB,A^2B,...,A^{n-1}B] = n$, and matrix A has only complex eigenvalues.

Theorem 8: Suppose the set U is a cone with vertex at zero and a nonempty interior in the space R^m. Then the dynamical system (1) is U-controllable from zero if and only if

1. $\mathrm{Rank}[B,AB,A^2B,...,A^{n-1}B] = n$.
2. There is no real eigenvector $v\in R^n$ of the matrix A^{tr} satisfying $v^{tr}Bw\leq 0$ for all $w\in U$.

For the single input system, that is, m=1, Theorem (7) reduces to the following corollary.

Corollary 8: Suppose that m=1 and U=[0,1]. Then the dynamical system (1) is U-controllable from zero if and only if it is controllable without any constraints; in other words, $\mathrm{rank}[B,AB,A^2B,...,A^{n-1}B] = n$, and matrix A has only complex eigenvalues.

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Biographical Sketch

Jerzy Klamka was born in Poland in 1944. He received M.Sc. and Ph.D. degrees in control engineering from the Silesian Technical University in Gliwice, Poland, in 1968 and 1974, respectively. He also received M.Sc. and Ph.D. degrees in mathematics from the Silesian University in Katowice, Poland, in 1971 and 1978, respectively. In 1981 he received habilitation in control engineering and in 1990 titular professor in control engineering from the Silesian Technical University in Gliwice.

Since 1968 he has been working for the Institute of Control Engineering of the Silesian technical University in Gliwice, where he is now a full professor. In 1973 and 1980 he taught semester courses in mathematical control theory at the Stefan Banach International Mathematical Center in Warsaw.

He has been a member of the American Mathematical Society (AMS) since 1976, and Polish Mathematical Society (PTM) since 1982. He is also a permanent reviewer for *Mathematical Reviews* (from 1976) and for *Zentralblatt für Mathematik* (from 1982). In 1981 and 1991 he was awarded the Polish Academy of Sciences awards. In 1978, 1982, and 1990 he received the awards of the Ministry of Education, and in 1994 he was awarded the Polish Mathematical Society award.

In 1991 he published the monograph *Controllability of Dynamical Systems*, (Kluwer Academic Publishers, Dordrecht, the Netherlands). In the last 30 years he has published more than 100 papers in international journals, for example, in: *IEEE Transactions on Automatic Control, Automatica, International Journal of Control, Journal of Mathematical Analysis and Applications, Systems and Control Letters, Foundations of Control Engineering, Systems Science, Kybernetika, IMA Journal on Mathematical Control and Information, Nonlinear Analysis, Theory, Methods and Applications, Systems Analysis, Modeling, Simulation, Archives of Control Science, Applied Mathematics and Computer Science, Advances in Systems Science and Applications, Bulletin of the Polish Academy of Sciences, Mathematical Population Dynamics, Lecture Notes in Control and Information Sciences, Analele Universitati din Timisoara*, and *Acta Mathematicae Silesiane.* He has taken part in many international Congresses, Conferences, and Symposiums.

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His major current interest is controllability theory for linear and nonlinear dynamical systems, and in particular controllability of distributed parameter systems, dynamical systems with delays, and multidimensional discrete systems.

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