CONTROLLABILITY AND OBSERVABILITY OF NONLINEAR SYSTEMS

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Summary

A survey of some of the main approaches for studying controllability and observability of nonlinear systems is given. Emphasis is on differential geometric tools like the Lie-bracket and Lie-derivative, which forms a natural starting point for extending standard results on linear controllability and observability to nonlinear systems.

1. Introduction

Two of the fundamental concepts appearing in the earlier studies on linear control systems are controllability and observability. In particular, it was through the work of R.E. Kalman that the notion of controllability of a linear system was shown to be of interest in itself, and not only as it was appearing in the context of optimal control. Linear controllability turns out to be of importance in many contexts, with perhaps as one of the most useful applications, feedback stabilization or rather pole placement through state feedback. On the other hand, controllability has a very simple and appealing formulation in that a linear control system is said to be controllable if for any given pair of initial and final states there exists an input function that 'steers' the system from initial to final state.

Also, observability in linear systems was developed simultaneously in the early 1960s, again under the impulses of R.E. Kalman. Besides the duality with linear controllability, like duality between a vector space and its dual, the importance of observability was recognized in estimation and state reconstruction problems. A linear system with a linear output map is said to be observable if for a given input function, the output map uniquely determines the (initial) state of the system.

Today, a wealth of information on controllability and observability of systems is available, and these concepts are now basic in a canonical description of linear systems. Starting in the early 1970s research has been directed also towards controllability and observability of nonlinear systems. Motivated in part by the linear theory, the aim was to develop similar results as there are available in the linear (time-invariant) setting. It soon turned out that this program might become too ambitious, in that apart from a few particular generalizations such as linear time-varying systems and bilinear control systems, a completely parallel theory on nonlinear controllability and observability is not feasible. Therefore various weaker notions of nonlinear controllability and observability have been developed in the 1970 and 1980s, all with an emphasis on computational characterizations, and their implications on the system structure.

The aim of this chapter is to give an insightful introduction to certain nonlinear controllability and observability notions. Just like for linear systems, both concepts have their application in the design of nonlinear controllers. We will not go into the details of these applications, but refer the reader to the chapters *Input-output Stability*, *Design of Nonlinear Control Systems*, *Feedback Linearization* and *Output Regulation* where some of these applications will become clear.

The organization of the chapter is as follows. In the next section, some preliminaries will be given. Section 3 treats the controllability and accessibility (a weaker form of controllability) of nonlinear control systems. Section 4 finally, treats the observability of nonlinear control systems.

2. Preliminaries

The nonlinear control systems that will be considered here are systems of the form

$$\begin{cases} \dot{x} &= f(x) + \sum_{i=1}^{m} u_i g_i(x) \\ y &= h(x) \end{cases} \qquad (1)$$

where $x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T \in \mathbb{R}^n$ are local coordinates for the smooth state space manifold *M*, $u = \begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix}^T \in U \subset \mathbb{R}^m$ are the controls, $y = \begin{pmatrix} y_1 & \cdots & y_p \end{pmatrix}^T \in \mathbb{R}^p$ are the outputs, $f, g_1, \cdots, g_m$ are smooth vector fields on *M*, and $h = \begin{pmatrix} h_1 & \cdots & h_p \end{pmatrix}^T : M \to \mathbb{R}^p$ is a smooth function. The vector field f is called the *drift vector field* and the vector fields $g_1, \cdots, g_m$ are called the *control vector fields*. The system (1) is called *driftless* if $f \equiv 0$.

The set of *admissible controls* $\mathcal{U}$ will be a subset of the functions from $\mathbb{R}^+ = [0,\infty)$ to *U*. We will assume that the sets *U* and $\mathcal{U}$ satisfy the following conditions.

Assumption 1.

(a) The input space *U* is such that the set of associated vector fields of the system (1), $\mathcal{F} = \{ f + \sum_{i=1}^{m} u_i g_i \mid \begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix}^T \in U \}$, contains the vector fields $f, g_1, \cdots, g_m$.

(b) $\mathcal{U}$ consists of the piecewise constant functions which are piecewise continuous from the right.

If the sets *U* and $\mathcal{U}$ satisfy the conditions in Assumption 1, it may be shown that for any initial condition $x(0) = x^0$ and any given admissible control function $u(\cdot) \in \mathcal{U}$, there exists a $0 < t_f \leq +\infty$ such that the solution of (1) is defined on $[0, t_f)$, and is unique on $[0, t_f)$. This solution will be denoted by $x(t, x^0, u)$, while the resulting output at time $t \in [0, t_f)$ will be denoted by $y(t, x^0, u)$.

In studying controllability and observability of nonlinear control systems, one needs the concepts of *Lie bracket of vector fields* and *Lie derivative of a function*. Below, we will give a definition of these concepts in local coordinates (see *Lie Bracket* for a coordinate free definition).

Definition 1. (Lie bracket and Lie derivative) Consider an *n*-dimensional smooth manifold *M* with local coordinates $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T$. Let X, Y be smooth vector fields on *M*, and let $\phi : M \to \mathbb{R}$ be a smooth function.

(a) The *Lie bracket* of the vector fields *X* and *Y*, which is denoted by $[X, Y]$, is a smooth vector field on *M*, which in local coordinates is given by

$$[X,Y](x) := \frac{\partial Y}{\partial x}(x) X(x) - \frac{\partial X}{\partial x}(x) Y(x) \tag{2}$$

where

$$\frac{\partial X}{\partial x}(x) := \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \cdots & \cdots & \frac{\partial X_1}{\partial x_n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial X_n}{\partial x_1} & \cdots & \cdots & \frac{\partial X_n}{\partial x_n} \end{pmatrix}(x)$$

and $\frac{\partial Y}{\partial x}(x)$ is defined analogously.

(b) The *Lie derivative* of ϕ along *X*, which is denoted by $\mathcal{L}_X\phi$, is a smooth real valued function on *M*, which in local coordinates is given by

$$\mathcal{L}_X\phi(x) := \sum_{i=1}^{n} X_i(x)\frac{\partial\phi}{\partial x_i}(x) \tag{3}$$

3. Controllability and Accessibility

In this section we will study controllability of nonlinear control systems, as well as a weaker form of controllability which is known as accessibility. Roughly speaking, a system is controllable if one can steer from any point $x^0 \in M$ to any other point $x^1 \in M$ by choosing an appropriate control function $u \in \mathcal{U}$. For nonlinear systems, however, controllability in this sense is a notion that is too difficult to check in general. Therefore, also more restricted (local) versions of controllability are defined for nonlinear systems. To introduce these notions, we first need to define what we mean by a reachable set for a nonlinear system.

Definition 2. (Reachable set) Consider the nonlinear control system (1), and let $x^0 \in M$ be given. Let $V \subset M$ be a neighborhood of x^0.

a) The set $R^V(x^0,T)$ is the set of all points that can be reached from x^0 at time $T > 0$, following trajectories which remain in *V* for $0 \le t \le T$, i.e.,

$$\begin{aligned} R^V(x^0,T) = \{x \in M \mid & \text{ there exists a } u(\cdot) \in \mathsf{U} \text{ such that} \\ & x(t,x^0,u) \in V \ (0 \le t \le T) \text{ and } x(T,x^0,u) = x\} \end{aligned} \tag{4}$$

b) The *V*-reachable set at time *T*, which is denoted by $R_T^V(x^0)$, is defined by

$$R_T^V(x^0) := \bigcup_{\tau \le T} R^V(x^0,T) \tag{5}$$

c) The reachable set from x^0, which is denoted by $R(x^0)$, is defined by

$$R(x^0) := \bigcup_{T>0} R^M(x^0,T) \tag{6}$$

Definition 3. (Controllability)

a) The nonlinear control system (1) is called *controllable* if for every $x^0 \in M$ one has that

$$R(x^0) = M \tag{7}$$

b) Let $x^0 \in M$ be given. Then the nonlinear control system (1) is called *locally controllable from* x^0 if for every neighborhood V of x^0 and every $T > 0$ one has that $R_T^V(x^0)$ contains a neighborhood of x^0.

3.1. Controllability and Linearization

In studying nonlinear control systems, a first approach is very often to consider its so called *linearization* around an equilibrium point. For a nonlinear control system of the form (1), this approach proceeds as follows. An *equilibrium point* $(x^0, u^0) \in M \times U$ is defined to be a point at which the system is at rest, i.e., $f(x^0) + g(x^0)u^0 = 0$. Now let an equilibrium point (x^0, u^0) for (1) be given, and define $x_\delta := x - x^0$, $u_\delta := u - u^0$, $y_\delta := y - h(x^0)$, and

$$A := \frac{\partial f}{\partial x}(x^0) + \sum_{i=1}^{m} u_i^0 \frac{\partial g_i}{\partial x}(x^0),\ B := \begin{pmatrix} g_1(x^0) & \cdots & g_m(x^0) \end{pmatrix},\ C := \frac{\partial h}{\partial x}(x^0) \tag{8}$$

Taking a Taylor series of (1) around (x^0, u^0), one then obtains $\dot{x}_\delta = Ax_\delta + Bu_\delta + \text{h.o.t.}$ and $y_\delta = Cx_\delta + \text{h.o.t.}$ where "h.o.t." stands for "higher order terms". Neglecting the higher order terms, one obtains the *linearization* of (1) around (x^0, u^0), which is given by

$$\begin{cases} \dot{X}_\delta = Ax_\delta + Bu_\delta \\ y_\delta = Cx_\delta \end{cases} \tag{9}$$

Since near the equilibrium the linearized system (9) is a first order approximation of the nonlinear control system (1), one might hope to be able to make statements about the controllability of (1) based on the controllability properties of the linearized system (9). This is partly possible, as becomes clear from the following result.

Theorem 1. *Consider the nonlinear control system* (1), *and let* (x^0, u^0) *be an equilibrium point. Assume that* U *contains a neighborhood of* u^0. *For* $\varepsilon > 0$, *define* $\mathcal{U}_\varepsilon := \{u(\cdot) \in \mathcal{U} \mid \|u(t) - u^0\| < \varepsilon\ (t \geq 0)\}$. *If the linearized system* (9) *is controllable then for every* $\varepsilon > 0$ *the system* (1) *is locally controllable from* x^0, *where the control functions* $u(\cdot)$ *are taken from the set* $\mathcal{U}_\varepsilon$.

Remark 1. Recall that the linearized system (9) is controllable if and only if the matrix

$$\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \tag{10}$$

has full rank (see *System Characteristics: Stability, Controllability, Observability*).

The usefulness of Theorem 1 is restricted. In the first place, it only gives information about controllability in the neighborhood of equilibria. Furthermore, it may be that a nonlinear system is controllable around an equilibrium point, while its linearization around the equilibrium is not controllable. This is illustrated by the following example.

Example 1. Consider the model of a car in Figure 1.

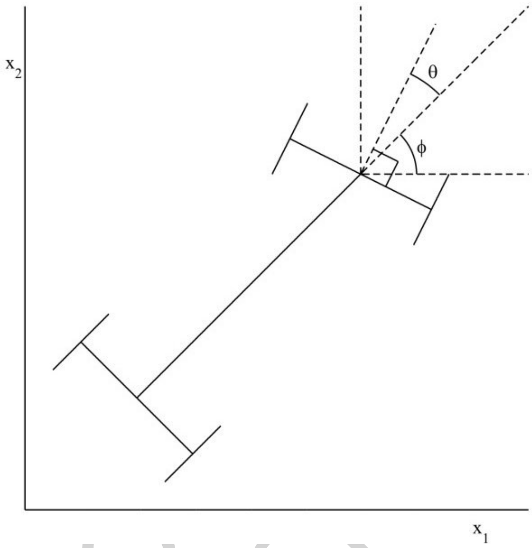

Figure 1. Model of a car

As state of the system we take $x = (x_1, x_2, \phi, \theta)$, where (x_1, x_2) are the Cartesian coordinates of the center of the front axle, ϕ is the angle between the horizontal and the longitudinal axle of the car, and θ is the angle between the longitudinal axis and the line perpendicular to the front axis. Further, as control u_1 we take the angular velocity of the front axis, and as control u_2 the forward velocity of the car. We then obtain the following model:

$$\begin{cases} \dot{x}_1 &= \cos(x_3 + x_4)u_2 \\ \dot{x}_2 &= \sin(x_3 + x_4)u_2 \\ \dot{x}_3 &= \sin(x_4)u_2 \\ \dot{x}_4 &= u_1 \end{cases} \tag{11}$$

The linearization of (11) around the origin is given by

$$\begin{cases} \dot{x}_{\delta 1} &= u_{\delta 2} \\ \dot{x}_{\delta 2} &= 0 \\ \dot{x}_{\delta 3} &= 0 \\ \dot{x}_{\delta 4} &= u_{\delta 1} \end{cases} \tag{12}$$

From Remark 1 it follows that (12) is not controllable. However, from experience we know that the system (11) should be controllable. This leads to the conclusion that for controlling (11) we cannot use the linearization (12). In the following subsection we will study this example further, and show that it is indeed controllable.

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Biographical Sketches

Henri Huijberts (1962) obtained his MSc-degree and PhD-degree in Applied Mathematics from the University of Twente, Enschede, The Netherlands in 1987 and 1991 respectively. From 1991 until 1999 he was a Lecturer in Nonlinear Dynamical Systems in the Department of Mathematics and Computing Science of Eindhoven University of Technology, Eindhoven, the Netherlands. Since then he has been affiliated with the Department of Engineering, Queen Mary, University of London, United Kingdom, where he is currently Reader in Control and Dynamics. He has published a considerable number of journal and conference papers in the area of nonlinear control and dynamics. Henri Huijberts has been an Associate Editor of the IEEE Control Systems Society Conference Editorial Board, SIAM Journal on Control on Optimization, and IEEE Transactions on Automatic Control. Currently, he is a Subject Editor for the International Journal of Robust and Nonlinear Control. He is a Fellow of the Institute of Mathematics and its Applications, and a Senior Member of the Institute of Electrical and Electronics Engineers Inc.

Henk Nijmeijer (1955) obtained his MSc-degree and PhD-degree in Mathematics from the University of Groningen, Groningen, the Netherlands, in 1979 and 1983 respectively. From 1983 until 2000 he was affiliated with the Department of Applied Mathematics of the University of Twente, Enschede, the Netherlands. Since 1997 he was also part-time affiliated with the Department of Mechanical Engineering of the Eindhoven University of Technology, Eindhoven, the Netherlands. Since 2000, he is working full-time in Eindhoven, and chairs the Dynamics and Control section. He has published a large number of journal and conference papers, and several books, including the ‘classical’ Nonlinear Dynamical Control Systems (Springer Verlag, 1990, co-author A.J.van der Schaft). Henk Nijmeijer is Editor-in-Chief of the Journal of Applied Mathematics, Corresponding Editor of the SIAM Journal on Control and Optimization, and Editorial Board Member of the International Journal of Control, Automatica, European Journal of Control, Journal of Dynamical Control Systems, SACTA, International Journal of Robust and Nonlinear Control, and the Journal of Applied Mathematics and Computer Science. He is a Fellow of the Institute of Electrical and Electronics Engineers Inc. and was awarded the IEE Heaviside premium in 1987.