WELL-POSEDNESS OF HYBRID SYSTEMS

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Keywords: Hybrid Systems, well-posedness, hybrid automata, supervisory control, differential inclusions, complementarity problems, complementarity systems, even/flow formulas, multi-modal systems, passive systems, piecewise linear systems, differential equations with discontinuous right-hand sides, Zeno behaviour, nonsmooth dynamical systems.

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Summary

Well-posedness problems arise in hybrid systems theory as a consequence of the use of implicit descriptions and of solution concepts that are based on relaxations. Examples show that the well-posedness issue is considerably more complex in hybrid systems than in continuous systems, as a result of a number of factors including the possible presence of sliding modes, the interaction of guards and invariants, and the occurrence of left or right accumulations of event times. Description formats that are based on implicit or relaxed specifications are typically connected to particular subclasses of hybrid systems, and so there is no general theory of well-posedness of hybrid systems; however, the questions that need to be answered are similar in each case. This chapter surveys several description formats and solution concepts that are used for hybrid systems. We concentrate on well-posedness in the sense of existence and uniqueness of solution, without requiring continuous dependence on initial conditions. A selection of results available in the literature is presented for the subclasses of multi-modal linear systems, complementarity systems, and differential equations with discontinuous right-hand sides.

1. Introduction

Very broadly speaking, scientific modeling may be defined as the process of finding common descriptions for groups of observed phenomena. Often, several description forms are possible. To take an example from not very recent technology, suppose we want to describe the flight of iron balls fired from a cannon. One description can be obtained by nothing that such balls approximately follow parabolas, which may be parameterized in terms of firing angle, cannon ball weight, and amount of gun powder used. Another possible description characterizes the trajectories of the canon balls as solutions of certain differential equations. The latter description may be viewed as being fairly *indirect;* after all it represents trajectories only as solutions to some problem rather than expressing directly what the trajectories are, as the first description form does. On the other hand, the description by means of differential equations is applicable to a wider range of phenomena, and one may therefore feel that it represents a deeper insight. Besides, interconnection (composition) becomes much easier since it is in general much easier to write down equations than to determine the solutions of the interconnected system.

There are many examples in science where, as above, an implicit description (that is, a description in terms of a mathematical problem that needs to be solved) is useful and possibly more powerful than explicit descriptions. Whenever an implicit description is used, however, one has to show that the description is a "good" one in the sense that the stated problem has a well-defined solution. This is essentially the issue of well-posedness.

In this chapter we are concerned with *hybrid dynamical systems*, that is, systems in which continuous dynamics and discrete transitions both occur and influence each other. Many different description formats have been proposed in recent years for such systems; some proposed forms are quite direct, others lead to rather indirect descriptions. The direct forms have advantages from the point of view of *analysis*, but the indirect forms are often preferable from the perspective of *modeling* (specification); examples will be seen below. The more indirect a description form is, the harder it becomes to show that solutions are well-defined. Below we discuss a number of results on existence and uniqueness of solutions for given initial conditions in the context of various description formats for hybrid systems. It should be noted that this is still a very active research area, and so what we present can be no more than an impression of the state of the art at the moment of this writing (summer 2001).

We consider here systems in which the description of continuous dynamics in based on ordinary differential equations; in particular, we do not consider delayed arguments, partial differential equations, or stochastic differential equations. All of these settings require their own notions of well-posedness. Even in the context of ordinary differential equations, there are situations in which one is naturally led to the consideration of well-posedness problems for systems with mixed boundary conditions (i.e. partly initial conditions, partly final conditions). Here however we shall concentrate on initial value problems. Furthermore we only consider models that are formulated in continuous time. Discrete-time models are often stated in explicit form so that well-posedness is not much of an issue; that is not to say, of course, that implicit discrete-time models would not be sometimes useful.

2. Model Classes

We begin by introducing a number of description formats of hybrid systems. As already noted above, many different formats have been proposed, and so we can present only a selection.

2.1. The Hybrid Automaton Model

Hybrid systems research is sometimes viewed as a merger between dynamical systems/control theory on one side and computer science/automata theory on the other. It is therefore natural to look for description forms that combine elements from both sides. One way is to start with models that are used in computer science and to extend these with elements from continuous systems theory.

In computer science, direct description forms appear to dominate. A typical specification of a finite automaton consists of a list of all states together with the transitions that may occur from each of these states and the conditions under which these transitions may take place. In more structured descriptions, such as Petri nets, the collection of states is not listed explicitly, but there is still for each state a simple rule that defines the possible successor states. *Determinism* (in the sense that a uniquely determined trajectory exists for a given initial condition and, if applicable, a given input sequence) is not always required; for instance if the model is to be used to prove a certain property and it is suspected that the proof will not depend on certain details of the dynamics, it is very convenient to leave these details unspecified. The discrete systems studied by computer scientists are often very large and so a key issue is *compositionality*, that is, the feasibility of putting subsystems together to form a larger system.

The hybrid automaton model as proposed by Alur *et al*. may be described briefly as follows. The discrete part of the dynamics is modeled by means of a graph whose vertices are called *locations* and whose edges are *transitions*. The continuous state takes values in a vector space X . To each location there is a set of trajectories, which are called *activities,* and which represent the continuous dynamics of the system. Interaction between the discrete dynamics and the continuous dynamics takes place through *invariants* and *transition relations*. Each location has an invariant associated to it, which describes the conditions that the continuous state has to satisfy at this location. Each transition has an associated transition relation, which describes the conditions on the continuous state under which that particular transition may take place and the effect that the transition will have on the continuous state. Invariants and transition relations play supplementary roles: whereas invariants described when a transition *must* take place (namely when otherwise the motion of the continuous state as described in the set of activities would lead to violation of the conditions given by the invariant), the transition relations serve as "enabling conditions" that describe when a particular transition *may* take place.

In the model, transitions are further equipped with *synchronization labels*, which express synchronization constraints between different automata. This construct allows the introduction of a notion of *parallel composition* between two automata. The component automata are assumed to have the same continuous state space, and the set of activities at each location of the composition (which is a pair of locations of the component automata) is the intersection of the sets of activities at the corresponding component locations.

Various ramifications of the hybrid automaton model have been proposed in the literature. Sometimes the notion of a transition relation is split up into two components, namely a *guard* which specifies the subset of the state space where a certain transition is enabled, and a *jump function* which is a (set-valued) function that specifies which new continuous states may occur given a particular transition and a particular previous continuous state. Often the hybrid automaton model is extended with a description format for continuous dynamics, typically systems of differential equations. Versions of the hybrid automaton model which include external inputs have been proposed in the literature.

2.2. Explicit State-space Model

Many studies in continuous-variable control theory are based on the model $\dot{x}(t) = f(x(t), u(t))$ where $x(t)$ denotes a continuous state variable and $u(t)$ is a continuous control variable. Often one just writes $\dot{x} = f(x,u)$, suppressing the dependence of all variables on time. A model in the same spirit for hybrid systems may be written down as follows:

$$\dot{x} = f(x, q, u, r) \tag{1}$$

$$q^{+} = g(x, q, u, r), \tag{2}$$

where x and u are continuous state and control variables as before, q and r denote discrete state and control variables, and superscript "+" is used to indicate "next state".

The function g express updates of the discrete state which depend on the current values of both the continuous and the discrete state, as well as on the continuous and discrete inputs.

We call the above model "explicit" even though the continuous dynamics is actually given in terms of a problem, to wit a differential equation, since the model gives the time derivative of the continuous state variable explicitly as a function of all variables in the system. The discrete-state update is given explicitly as well. For such models, the well-posedness issue is rather easy (if not trivial) because of the explicit nature.

2.3. Supervisor Model

One of the sources of interest in hybrid systems is *supervisory control*, that is, control of a continuous system by means of a discrete system. (Other names for such control schemes are in use as well, for instance *intelligent control*.) In the context of supervisory control, it is natural to use a framework in which there is a continuous system on the one hand, a discrete system on the other, and the two are connected in a feedback loop which involves a translation from digital to analog signals and back. The continuous dynamics can be considered to be parameterized by the discrete state of the supervisor. The continuous state is discretely monitored, typically through a partition of the state space, and the resulting signal is processed by the supervisor.

2.4. Differential Inclusions

During the past decades, extensive studies have been made of *differential equations with discontinuous right hand sides*. For a typical example, consider the following specification:

$$\dot{x} = f_1(x) \quad (h(x) > 0) \tag{3a}$$

$$\dot{x} = f_2(x) \quad (h(x) < 0), \tag{3b}$$

where h is a real-valued function. A system of this form can be looked at either as a discontinuous dynamical system or as a hybrid system of a particular form. The specification above is obviously incomplete since no statement is made about the situation in which $h(x) = 0$. One way to arrive at a solution concept is to adopt a suitable *relaxation*. Specifically, in a *convex* relaxation one would rewrite Eq.(3) as

$$\dot{x} \in F(x), \tag{4}$$

where the set-valued function $F(x)$ is defined by

$$F(x) = \{f_1(x)\} \;\; (h(x) > 0), \quad F(x) = \{f_2(x)\} \;\; (h(x) < 0),$$

$$F(x) = \{y \mid \exists a \in [0,1] \text{s.t. } y = af_1(x) + (1-a)f_2(x)\} \quad (h(x) = 0), \tag{5}$$

where it is assumed (for simplicity) that f_1 and f_2 are given as continuous function defined on $\{x \mid h(x) \geq 0\}$ and $\{x \mid h(x) \leq 0\}$ respectively. The discontinuous dynamical system has not been reformulated as a *differential inclusion,* and so solution concepts and well-posedness results can be applied that have been developed for systems of this type.

2.5. Complementarity Systems

Systems of the form (3) are sometimes known as *variable-structure systems;* they describe a type of mode-switching. A similar mode-switching behaviour is obtained from a class of systems known as *complementarity systems*. Equations for a Complementarity system may be written in terms of a state variable x and auxiliary variables v and z , which must be vectors of the same length. Typical equations are:

$$\dot{x} = f(x,v) \tag{6a}$$

$$z = h(x,v) \tag{6b}$$

$$0 \leq z \perp v \geq 0\,, \tag{6c}$$

where the last line means that the components of the auxiliary variables v and z should be nonnegative, and that for each index i and for each time t at least one of the two variables $v_i(t)$ and $z_i(t)$ should be equal to 0. Variables that satisfy such relations occur naturally in various problems; think of current/ voltage in connection with ideal diodes, flow/pressure in connection with one-sided valves, Lagrange multiplier / slack variable in optimization subject to inequality constraints, and so on. Like (3), the system (6) consists of a number of different dynamical systems or “modes” that are glued together. The modes can be thought of as discrete states. They correspond to a fixed choice, for each of the indices i , between the two possibilities $v_i \geq 0,\ z_i = 0$ and $v_i = 0,\ z_i \geq 0$, so that a Complementarity system in which the vectors v and z have length m has 2^m different modes. The specification (6) is in general not complete yet; one has to add a rule that describes possible jumps of state variable x when a transition from one mode to another takes place.

The description (6) is implicit in the discrete variables. Suppose we are at a point where a transition must occur because otherwise an inequality constraint would be violated. There may or may not be a unique mode in which the differential equations of (6a), together with the equality constraints in (6c) that are implied by the given mode, produce a solution that satisfies the complementary inequality constraints in (6c) at least for some positive time interval. If there is indeed a unique solution to this problem, then this mode is taken as the successor state. In case this procedure can be successfully carried out at all points of the continuous state space, the Complementarity system can in principle be rewritten in the explicit hybrid automaton format., but the representation that is obtained may be very awkward.

2.6. Event/Flow Formulas

The notion of time as used in hybrid systems theory is different both from the time used in continuous system theory and from the time used in automata theory. In the theory of continues dynamical systems, typically the real line or intervals of it are taken as the time axis, whereas in automata theory time just serves to impose an order on separate events, so that an appropriate model of time is often provided by (a subset of) the integers. In hybrid systems theory, time could be described as a “punctured real axis”. One may even consider events of higher multiplicity, so that the time points at which events take place may have different integer weights. In systems that are obtained by composition of smaller subsystems, here may be events in some subsystems that are not shared by other subsystems and so one may argue that actually the time axes for different subsystems should be different. A description format for hybrid systems that incorporates a notion of “multitime” has been proposed by Van der Schaft and Schumacher. The format allows specification of equations of a fairly general type in which “flow conditions” (such as differential equations) and “event conditions” (such as transition rules) can be specified. In general the format is implicit both in the continuous variables and in the discrete variables.

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