OPTIMAL CONTROL OF HYBRID SYSTEMS

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Summary

The theory of optimal control deals with the problem to find a trajectory of a dynamical system that minimizes a given cost function. The problem has a long history, both for systems with purely continuous dynamics and for purely discrete systems. An application of the first category is to find a path for a space rocket that minimizes fuel consumption. A problem of the second category is to find the shortest path through a graph.

Fundamental contributions to mathematics as well as engineering have been made through the theory of optimal control. It is therefore natural to ask to what extent the concepts and results can be extended to hybrid systems, i.e. systems with interaction between continuous dynamics and discrete events.

There are two main approaches to optimal control in a continuous state space, commonly referred to as "dynamic programming" and "the maximum principle". The first approach describes the optimal cost as function of the initial state, while the second is devoted properties of an optimal trajectory. This chapter is mainly focused on dynamic programming, since this theory is also well developed for discrete systems and the hybrid case therefore fits nicely in between.

1. Introduction

Hybrid systems are systems that contain interaction between continuous and discrete dynamics. Such systems are common in engineering, for example when technical equipment with discrete dynamics interacts with a physical environment that evolves with continuous dynamics.

An important role of optimal control is to provide a theoretical foundation for synthesis problems in engineering and science. The synthesis objectives are specified in an optimization criterion, or a cost/performance function. This is a mathematical expression that should be minimized/maximized subject to given constraints. The cost function is used to penalize various quantities such as energy consumption, deviation from a desired set point, unsafe states, etc. Once dynamics, constraints and cost function have been specified for the problem, the synthesis task has been converted into a mathematical problem, ready for analysis and computational treatment.

This chapter will explain some basic ideas in optimal control of hybrid systems. The idea of dynamic programming is first described for purely discrete systems and then for purely continuous ones, before dealing with a general class of hybrid systems. The method is applied to a gear shift problem. Finally, we consider some other aspects of optimal control, including a hybrid version of the maximum principle.

2. Hybrid Dynamic Programming

Dynamic Programming (DP) was introduced by Bellman in 1957, applying Hamilton-Jacobi optimization theory to problems of control. The basic idea, sometimes called the principle of optimality, is that every optimal trajectory can be split into two pieces, each of which is optimal in a certain sense. The optimal cost for the original problem is equal to the sum of the optimal costs for the two pieces. In continuous time, the so called Hamilton-Jacobi-Bellman (HJB) equation is obtained in the limit as the length of one of the pieces approaches zero. Dynamic Programming naturally leads to a solution in the form of a “feedback law”.

Before considering the general case of hybrid systems, the main steps will be reviewed for discrete and continuous systems separately.

2.1. Dynamic Programming in Discrete Systems

Consider the system

$$q(k+1) = \phi(q(k), \mu(k)) \ , \qquad q(0) = q_0 \ ,$$

where $q(k) \in Q$ is the state at time k and $\mu(k) \in U$ is the value of the control signal. Given a non-negative function $s : Q \times U \to \mathbf{R}$ with $s(q, \mu) > 0$ for $q \neq 0$, the problem is to find the optimal value function

$$V^*(q_0) = \inf_{\mu} \left[\sum_{k=0}^{\infty} s(q(k), \mu(k)) \right]$$

and corresponding minimizing input sequences $\mu(0), \mu(1), \ldots$.

Under general assumptions, the optimal value function is characterized by "Bellman's equation":

$$V^*(q) = \min_{\mu \in U}\{V^*(\phi(q,\mu)) + s(q,\mu)\}$$

There are several approaches for solution of this equation. One is to use "value iteration", where a sequence of functions $V_0(q), V_1(q), V_2(q), \ldots$ is generated by repeatedly evaluating the right hand side of Bellman's equation with V^* replaced by V_k. Another approach is to consider the problem

$$\begin{array}{lll} \text{Maximize} & V(q_0) & (1) \\ \text{subject to} & \begin{cases} V(q) \le V(\phi(q,\mu)) + s(q,\mu) & \text{for all } q,\mu \\ V(0) = 0 \end{cases} & \end{array}$$

which is solved by V^*. In fact, every solution to the constraints gives a lower bound on the optimal value function for all q and maximization yields the exact value at q_0. For a discrete system, where Q and U are finite, (1) is a linear programming problem for which efficient algorithms are available.

Example 1 Consider the discrete transportation problem illustrated in Figure 1. Such problems have been studied extensively since the 1940's. The cost for shipping some product between node i and node j is given by the number s_{ij}. The objective is to minimize the total cost for shipping the production in node 3 to consumers in node 0.

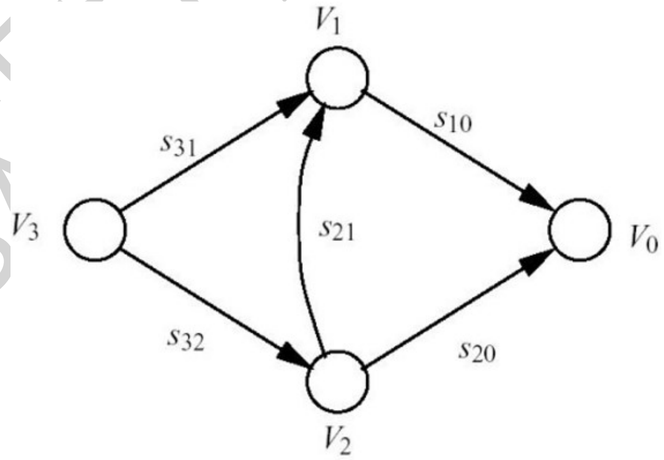

Figure1: The cost for transportation from node i to node j is s_{ij}. The production in node 3 should be transported to the consumer in node 0 while minimizing the transportation cost.

For this problem, the linear programming problem (1) takes the form

$$
\begin{array}{lll}
\text{Maximize} & V_3 & \\
\text{subject to} & V_3 - V_1 \leq s_{31} & V_0 = 0 \\
 & V_3 - V_2 \leq s_{32} & \\
 & V_2 - V_1 \leq s_{21} & \\
 & V_1 - V_0 \leq s_{10} & \\
 & V_2 - V_0 \leq s_{20} &
\end{array}
$$

Note that there is one variable V_i for each node and one inequality constraint for each path connecting two nodes. For every solution to the inequality constraints, the number V_3 provides a lower bound on the cost for shipping products from node 3 to node 0.

2.2. Dynamic Programming in Continuous Systems

Consider the system

$$\dot{x}(t) = f(x(t), u(t)) \quad , \qquad x(0) = x_0 ,$$

where $x(t) \in \mathbf{R}^n$ is the state at time t and $u(t) \in U$ is the value of the control signal. Given a function $l : \mathbf{R}^n \times U \to \mathbf{R}$, strictly positive except at $x = 0$, the problem is to find the optimal value function

$$V^*(x_0) = \inf_u \left[\int_0^\infty l(x(t), u(t)) dt \right]$$

and corresponding minimizing input signals u.

The analog of the discrete Bellman equation is the "Hamilton-Jacobi-Bellman equation":

$$\inf_u \left[\frac{\partial V^*}{\partial x} f(x,u) + l(x,u) \right] = 0$$

There is a rich literature devoted to this equation. One issue is the interpretation of the expression $\partial V^* / \partial x$, since the optimal value function V^* often may not be differentiable in the classical sense. The notion of "viscosity solution" has instead proved to be very useful.

Value iteration has no obvious counterpart in continuous time, but the linear programming approach does:

Maximize $\quad V(x_0)$

Subject to
$$\begin{cases} \frac{\partial V}{\partial x} f(x,u) + l(x,u) > 0 \text{ for all } x,u \\ V(0) = 0 \end{cases}$$

Under appropriate assumptions the optimal value is equal to $V^*(x_0)$, even if the optimization is restricted to differentiable functions *V*. Both the objective and the constraints are linear in *V*, but the problem is infinite-dimensional so computations are generally non-trivial.

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Biographical Sketches

Anders Rantzer was born in 1963. He received a Ph.D. degree in optimization and systems theory from the Royal Institute of Technology (KTH), Stockholm, Sweden. After postdoctoral positions at KTH and at IMA, University of Minnesota, he joined the Department of Automatic Control, Lund, in 1993. In 1999, he was appointed as professor of Automatic Control.

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