

## THE MOTION OF CELESTIAL BODIES

**Kaare Aksnes**

*Institute of Theoretical Astrophysics University of Oslo*

**Keywords:** celestial mechanics, two-body orbits, three-body orbits, perturbations, tides, non-gravitational forces, planets, satellites, space probes

### Contents

1. Introduction
  2. Two-body problem
    - 2.1. Orbit in Space
    - 2.2. Determination of Orbits from Observations
  3. Three-body problem
  4. Perturbations of planets and satellites
  5. Dynamics of asteroids and planetary rings
  6. Dynamics of artificial satellites and space probes
  7. Tides
  8. Non-gravitational perturbations
  9. Dynamical evolution and stability of the Solar System
  10. Conclusion
- Acknowledgement  
Glossary  
Bibliography  
Biographical Sketch

### Summary

The history of celestial mechanics is first briefly surveyed, identifying the major contributors and their contributions. The Ptolemaic and Copernican world models, Kepler's laws of planetary motion and Newton's laws of universal gravity are presented. It is shown that the orbit of a body moving under the gravitational attraction of another body can be represented by a conic section. The six orbital elements are defined, and it is indicated how they can be determined from observed positions of the body on the sky. Some special cases, permitting exact solutions of the motion of three gravitating bodies, are also treated. With two-body motion as a first approximation, the perturbing effects of other bodies are next derived and applied to the motions of planets, satellites, asteroids and ring particles. The main effects of the Earth's oblateness on the motions of artificial satellites are explained, and trajectories for sending a space probe from one planet to another are shown. The influences of gravitational tides and non-gravitational forces due to solar radiation and gas drag are also treated. Finally, the long-term evolution and stability of the Solar System are briefly discussed.

### 1. Introduction

Before turning to more technical aspects of celestial motions, a short historical survey

will be offered. The celestial bodies and their motions must always have attracted the attention of observant people. The Sun, Moon and planets were in many early cultures associated with gods. The names of our planets all come from Greco-Roman mythology. Comets were often viewed as bad omens of war, pestilence and disaster. The regular interval between new moons provided a basis for the first lunar calendars, while solar calendars were based on the apparent motion of the Sun among the stars, as manifested by changes in the times of rising and setting of these objects throughout the year.

Already around 700 B.C. the Babylonians had recorded on stone tablets the motions of the Sun, Moon and the planets against the star background, and had predicted lunar and solar eclipses with remarkable precision. But they don't seem to have had a geometrical picture of the motions in terms of *orbits*. The concept of orbits, or rather a device to compute orbits, was introduced by the Greek, notably Claudius Ptolemy. He developed in the *Almagest* around 140 A.D. the Ptolemaic system in which the Sun, Moon and planets each move in a circle (*epicycle*) whose center moves on the periphery of another circle (*deferent*) which is in turn centered on a point slightly displaced from the Earth's center. This *geocentric* world picture stood the ground for 1400 years until Nicolaus Copernicus on his death bed in 1543 introduced the *heliocentric* system with the Sun in the middle, but it took more than 100 years before the Copernican system was generally accepted.

The next fundamental progress was made by Johannes Kepler who in 1609 and 1619 published his three laws of planetary motion:

1. Each planetary orbit is an ellipse with the Sun situated at one of the foci.
2. The line joining the Sun and a planet will sweep out equal areas in equal times.
3. The cubes of the semi-major axes of the orbits are proportional to the squares of the planets' periods of revolution.

These laws and also Galileo Galilei's studies of the motions of falling objects, at about the same time, provided the basis for Isaac Newton's laws of motion and universal gravity, published in his famous *Philosophiae Naturalis Principia Mathematica* in 1687:

1. Every body remains at rest or uniform motion unless acted upon by an external force.
2. Acceleration is proportional to the impressed force and takes place in the direction in which the force acts.
3. To every action there is an equal and opposite reaction.
4. Two point masses will attract each other with a force which is proportional to the product of the masses divided by the square of the distance between them, the proportionality factor being called the universal constant of gravitation.
5. The gravitational attraction of an extended body of spherical shape will be as if all its mass had been concentrated at the center of the body.

This heralded the beginning of celestial mechanics which made it possible to accurately predict the motions of the celestial bodies. Newton showed that the gravitational attraction between two point masses, or finite bodies of spherical shape, leads to motions in orbits which are conic sections (see Figure 1). During the next 200 years this field was dominated by mathematicians and astronomers like L. Euler (1707-1783), J. L. Lagrange (1736-1812), P. S. Laplace (1749-1827), U. J. Leverrier (1811-1877), F. Tisserand (1845-1896) and H. Poincaré (1854-1912). Laplace had a very deterministic view of the Solar System, believing that if the positions and velocities of the celestial bodies could be specified at some instant, the motions at any time before or after this instant could be calculated with high accuracy. Lagrange showed that there are five possible equilibrium solutions of the three-body problem. In a failed attempt to find a general solution to this problem, Poincaré proved that the motions of three or more gravitating bodies are in general *chaotic*, meaning that the motions are unpredictable over very long time spans, thus negating Laplace's optimistic view. As we will comment on later, the consequences of chaotic motion have not been fully appreciated until recently.

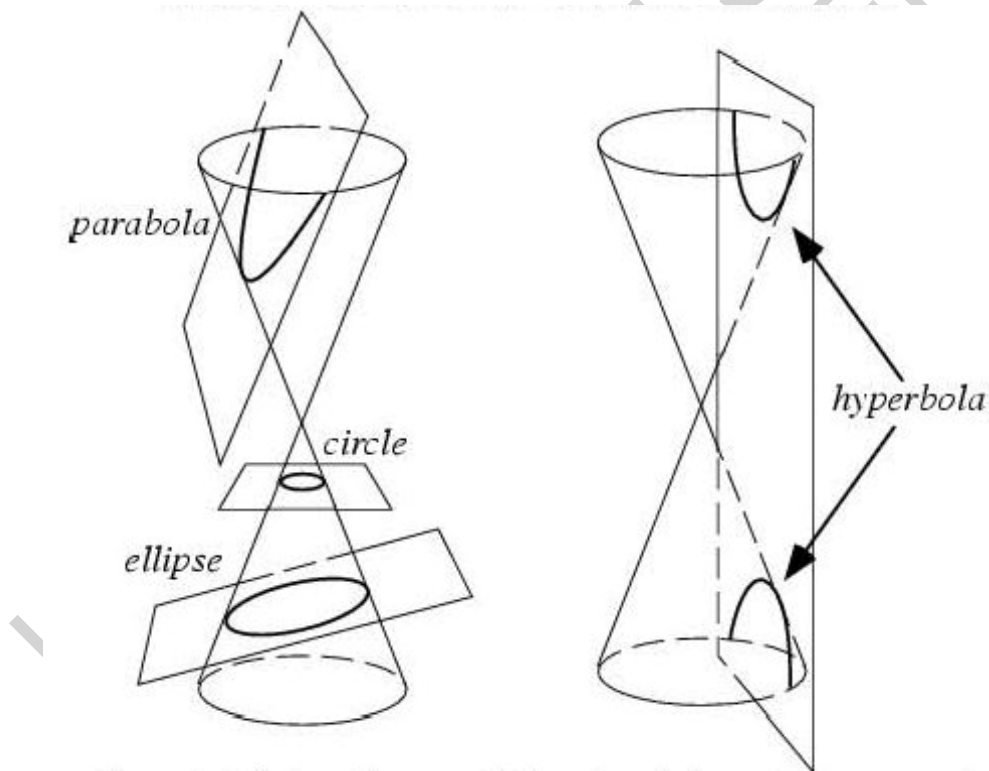


Figure 1. A circle, ellipse, parabola or hyperbola can be constructed as intersections of planes with a cone.

The introduction of the theory of special and general *relativity* by A. Einstein (1879-1955) has not affected planetary orbit theories appreciably, except for Mercury which moves close enough to the Sun, so that the deviation from Newtonian mechanics becomes measurable and was, in fact, the first confirmation of general relativity. Relativistic effects must also sometimes be taken into account for artificial satellites and space probes tracked with very high accuracy, and relativistic effects can be very pronounced for tight and massive binary stars.

In the first half of the twentieth century celestial mechanics went into a decline for lack of new and interesting problems. The launch of the first artificial satellite on 4 October 1957 marked the beginning of a new and exciting era in celestial mechanics. The advent of satellites, and later space probes, brought with it a demand for not only orbit calculation of unprecedented accuracy, but also *control* of a variety of orbit types not manifested by the natural bodies. A plethora of new satellites, asteroids, comets and planetary rings have been discovered with planetary probes and powerful modern telescopes on the ground and in space. The dynamics of planetary rings and the evolution of the Solar System are presenting special challenges to celestial mechanics.

## 2. Two-Body Problem

Let  $m_1$  and  $m_2$  be two point masses (or extended bodies of spherical shape with radial distribution of mass), a distance  $r$  apart, which attract each other with the force

$$F = G \frac{m_1 m_2}{r^2}$$

where  $G$  is the universal constant of gravitation. If  $\vec{r}_1$  and  $\vec{r}_2$  are the position vectors of the two masses measured from an arbitrary but unaccelerated point, then according to Newton's second law

$$m_1 \ddot{\vec{r}}_1 = G m_1 m_2 \frac{\vec{r}}{r^3}, \quad m_2 \ddot{\vec{r}}_2 = -G m_1 m_2 \frac{\vec{r}}{r^3}$$

where  $\vec{r} = \vec{r}_2 - \vec{r}_1$  is the position vector from  $m_1$  to  $m_2$ , and a dot above a symbol means differentiation with respect to time. This gives the differential equation called the *equation of motion* of the two-body problem:

$$\ddot{\vec{r}} + \mu \frac{\vec{r}}{r^3} = 0$$

where  $\mu = G(m_1 + m_2)$ . Newton showed that the resulting motion, i.e., the solution of this equation, can be expressed in polar coordinates  $(r, f)$  as

$$r = \frac{p}{1 + e \cos f}$$

which is the equation of a *conic section*. As illustrated on Figure 1, the intersection of a cone with a plane parallel to the base of the cone is a *circle* ( $e = 0$ ). By tilting the plane the circle turns into *ellipses* ( $0 < e < 1$ ) of increasing eccentricities until the plane is parallel to the side of the cone when a *parabola* ( $e = 1$ ) is cut out. If the tilt is further

increased *hyperbolas* ( $e > 1$ ) result. The parameter  $p$  (*semilatus rectum*) is the length of a normal to the symmetry axis of the ellipse, parabola or hyperbola through the focus of each of these curves. The angle  $f$  is called the *true anomaly* and is the angle between the radius vector  $r$  and the direction to the nearest point on the curve (*pericenter*), see Figure 2. For an ellipse,  $p$  is related to the *semimajor axis*  $a$  and the *eccentricity*  $e$  through the equation  $p = a(1 - e^2)$ .

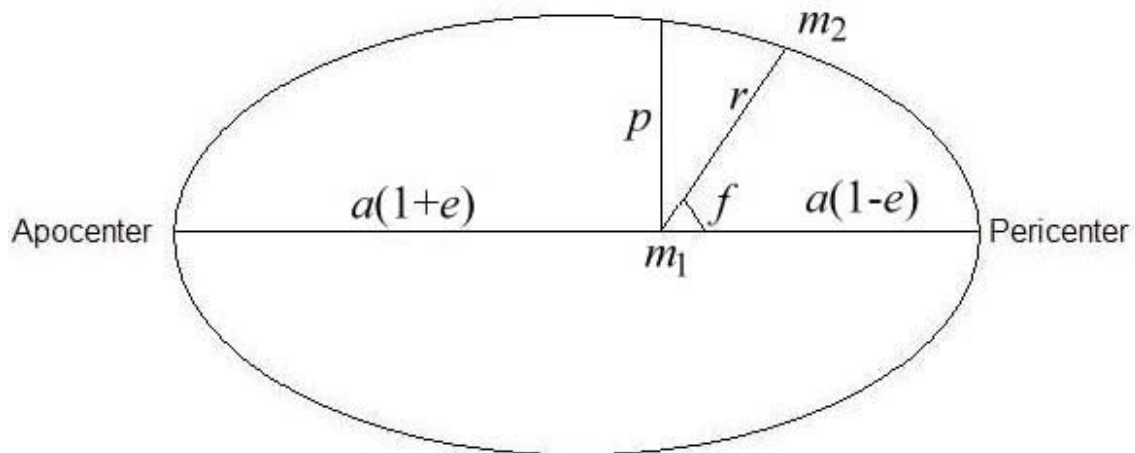


Figure 2. The position of  $m_2$  relative to  $m_1$  in an ellipse.

With these definitions, Kepler's second and third law can be expressed as

$$r^2 \dot{f} = \sqrt{\mu p}, \quad \frac{a^3}{P^2} = \frac{\mu}{(2\pi)^2}.$$

For a planet  $\mu = G(m_1 + m_2)$  where  $m_1$  is the mass of the Sun and  $m_2$  is the much smaller mass of the planet. Thus,  $\mu$  will vary slightly from planet to planet, whereas Kepler assumed that the right-hand side of the last equation is the *same* constant for all the planets. But even for massive Jupiter the deviation is only 1%, which was far too small to be detected in the observations that Kepler had at hand.

An important application of Kepler's third law has been to 'weigh' the planets which have satellites. Let the period, semi-major axis and mass be  $P, a, m$  for a planet and  $P', a', m'$  for its satellite. By applying the law, first to the planet in its motion about the Sun of mass  $M$ , and then to the satellite in its motion about the planet, we find after dividing the latter equation by the former:

$$\frac{P^2 a^3}{P'^2 a'^3} = \frac{m + m'}{M + m} \approx \frac{m}{M},$$

where the last approximation is based on the fact that the Sun's mass is much larger

than the planet's mass, which is in turn much larger than the satellite's mass. Since all the parameters on the left of this equation can be measured, this gives the planet's mass in terms of the Sun's mass.

The velocity  $V$  in an elliptic, parabolic or hyperbolic orbit is given by the *energy integral*

$$\frac{1}{2}V^2 - \frac{\mu}{r} = -\frac{\mu}{2a},$$

where  $a$  is positive for an ellipse, negative for a hyperbola and infinite for a parabola. The three terms of this equation represent, respectively, the kinetic energy, the potential energy and the total energy per unit mass. From this equation we obtain the *velocity of escape*  $V_E$  from the central body at the distance  $r$ , i.e. the parabolic velocity when  $a = \infty$ ,

$$V_E = \sqrt{\frac{2\mu}{r}} = \sqrt{2}V_C,$$

where  $V_C$  is the velocity in a circular orbit of radius  $r$ .

In order to calculate the true anomaly  $f$  for a given instant  $t$ , we need to introduce two auxiliary angles, which for objects in elliptic orbits are called the *mean anomaly*  $M$  and the *eccentric anomaly*  $E$ , related through *Kepler's equation*,

$$M = n(t - \tau) = E - e \sin E, \quad n = \sqrt{\frac{\mu}{a^3}}$$

where  $\tau$  is the time when the object passes through the pericenter and  $n$  is called the *mean motion*. After obtaining  $E$  from this equation (by iteration),  $f$  can be calculated from the equation

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{1+e}{1-e}} \tan\left(\frac{E}{2}\right).$$

## 2.1. Orbit in Space

Two-body motion can be specified through six constant *orbital elements*  $a$ ,  $e$ ,  $\tau$ ,  $I$ ,  $\Omega$ ,  $\omega$  of which the first three, as we have shown, define the size and shape of the orbit, and the last three orient the orbital plane in space (see Figure 3) with respect to some reference plane. These orbital elements are called

$a$  – the *semimajor axis*

$e$  – the eccentricity

$\tau$  – the time of pericenter passage

$I$  – the inclination (angle between orbital plane and reference plane)

$\Omega$ —the longitude of the ascending node (measured from a fixed direction in the reference plane)

$\omega$ —the argument of pericenter (angle from ascending node to pericenter)

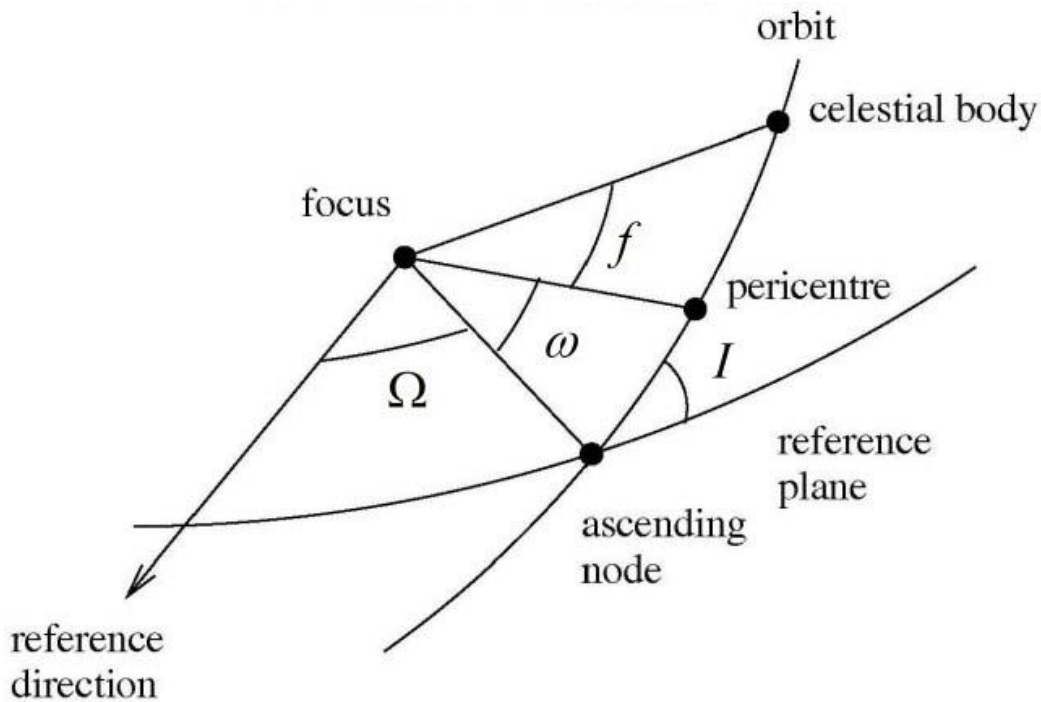


Figure 3. The orientation of the orbit of a celestial body by means of the angles  $I$ ,  $\Omega$ ,  $\omega$ .

## 2.2. Determination of Orbits from Observations

So far we have discussed how to compute positions and velocities of celestial bodies at specific times given their orbital elements. The inverse problem is considerably more complex and can only be solved by iteration. Here we will limit ourselves to a sketch of how to proceed. The classical methods for orbit determination assume that at least three positions on the sky at different times have been observed of the body whose orbit is sought. Modern observations of natural and artificial celestial bodies also include distance and radial velocity measurements with, for instance, radars and lasers. In the classical orbit determination problem, we assume that three sky positions are available. The intervals between the observations need to be neither too short nor too long for the methods to converge. The body's distance is unknown, so each observation yields only two coordinates. Three positions then supply 6 coordinates which are usually sufficient for determining the 6 orbital elements of the body. If we are dealing with a body in orbit about the Sun, the iteration starts off by guessing the distance of the body from the observer at the time of the middle observation. The heliocentric position of the Earth and the observer's position on the Earth are assumed known. By means of relations for

two-body motion, the heliocentric and geocentric positions at all three times are computed. The difference between the guessed and computed values of the distance, or of some related quantity, for the middle observation will make it possible to improve on the guess and repeat the iteration until convergence.

Normally more than three observations will be available. The orbit derived from three of the observations can then be improved by means of the method of *least-squares* in which the orbital elements are adjusted in such a way that the sum of squares of the residuals between observed and computed positions is minimized.

-  
-  
-

TO ACCESS ALL THE 27 PAGES OF THIS CHAPTER,  
Visit: <http://www.eolss.net/Eolss-sampleAllChapter.aspx>

### Bibliography

Goldreich P. & Tremaine S. (1979a). The formation of the Cassini Division in Saturn's rings, *Icarus* **34**, 240-253. [This studies how Mimas removes ring particles from the location of the 2:1 orbital resonance with Mimas].

Goldreich P. & Tremaine S. (1979b). Precession of the epsilon ring of Uranus, *Astronomical Journal* **84**, 1638-1641. [It is shown that Uranus' elliptical epsilon ring of particles rotates as one body].

Laskar J. (1989) A numerical experiment on the chaotic behavior of the Solar System. *Nature* **338**, 237-238. [Laskar shows that despite the chaotic nature of the motion of the planets, their orbits are stable for several hundred million years].

Malhotra, R. (1993). The origin of Pluto's peculiar orbit. *Nature*, **365**, 819-821. [This discusses why Pluto never collides with Neptune although their orbits intersect].

Morbidelli, A. (2002). Modern integrations of solar system dynamics. *Annual Review of Earth and Planetary Sciences*, **30**, 89-112. [The long-term development of planetary orbits is studied by means of new numerical integration techniques].

Murray C.D. and Dermott S.F. (1999). *Solar System Dynamics*, 592 pp, Cambridge University Press. [Modern textbook on celestial mechanics].

Peale, S.J. (1993). Pluto's strange orbit. *Nature* **365**, 788-789. [The consequences of Pluto's 2:3 orbital reference with Neptune is discussed].

Peale S.J., Cassen P. & Reynolds R.T. (1979) Melting of Io by tidal dissipation, *Science* **203**, 892-894. [this paper predicted Io's volcanism before it was discovered by the Voyager 1 spacecraft].

Roy A.E. (2005). *Orbital Motion*, 526 pp, Institute of Physics Publishing, ISBN 0 7503 10154. ISBN 0 521 57597 4. [Textbook on celestial mechanics].

Tsiganis K., Gomes R., Morbidelli A. & Levison H.F. (2005). Origin of the orbital architecture of the giant planets of the Solar System. *Nature* **435**, 459-461. [This simulation shows that Saturn, Uranus and Neptune formed much closer to the Sun than their current locations].

Wisdom J. (1982). The origin of the Kirkwood gaps: A mapping for asteroidal motion near the 3/1 commensurability. *Astronomical Journal* **87**, 557-593. [This paper explains the absence of asteroids with



periods  $1/3$  of Jupiter's period].

Wisdom J. and Holman M.J. (1991). Symplectic maps for the n-body problem, *Astronomical Journal* **102**, 1528-1538. [This paper introduces a numerical integration method allowing very long time steps].

### **Biographical Sketch**

**Kaare Aksnes**, born at Kvam, Norway 25 March 1938, Professor of Astronomy, University of Oslo, Norway, Masters Degree in astronomy, University of Oslo, Norway 1963, Ph.D. in astronomy, Yale University, New Haven, USA 1969.

He has for over 40 years worked on problems of dynamics of artificial and natural satellites, comets and asteroids, and has been involved with several US and European space missions (Mariner 9, Voyager 1&2, GPS, ERS, Rosetta). He headed for 16 years a Working Group for Planetary System Nomenclature for the International Astronomical Union. He was employed for 11 years in USA at the Smithsonian Astrophysical Observatory and Jet Propulsion Laboratory, and 30 years in Norway at the Norwegian Defence Research Establishment, University of Tromsø and University of Oslo. He has published more than 130 scientific papers.

Prof. Aksnes is a member of American Astronomical Society, International Astronomical Union and Norwegian Academy of Science and Letters.