CLASSICAL HAMILTONIAN PERTURBATION THEORY

Mikhail B. Sevryuk
V. L. Talroze Institute of Energy Problems of Chemical Physics, Moscow, Russia

Keywords: Hamiltonian systems, perturbations, invariant tori, conditionally periodic motions, quasi-periodic motions, resonances, Liouville–Arnold theorem, completely integrable systems, partially integrable systems, nearly integrable systems, action-angle variables, averaging, elimination of the "fast angles", formal power series, KAM theory, Kolmogorov tori, Whitney smooth families, resonant zones, nondegeneracy and nonresonance conditions, Nekhoroshev theory, stability, stickiness, Arnold diffusion

Contents

Summary

The classical Hamiltonian perturbation theory (the name "canonical perturbation theory" is also used) is a bridge between the general Hamiltonian dynamics and Celestial Mechanics. The word "classical" here means of course "finite dimensional, not quantum mechanical, not in the generalized sense, and not belonging to the realm of Statistical Mechanics" (in particular, we regard Celestial Mechanics as a part of Classical Mechanics). In this chapter, we outline some basic topics in the Hamiltonian perturbation theory from a general viewpoint. We do intentionally not describe any applications because many illuminative examples are presented in other chapters of this volume.

After two introductory Sections 1 and 2, we proceed to integrable Hamiltonian systems (Section 3) which are usually treated as "unperturbed" ones. The formal computational procedures of the theory are dealt with in Section 4; some of them were well developed by the end of the 19th century. The main contribution of the 20th century to the subject in question, the so-called Kolmogorov–Arnold–Moser (KAM) theory which allows one to explore non-integrable perturbations on a dynamical and geometric level rather than on a formal and computational one, is sketched in Section 5. We continue reviewing perturbed dynamics in Section 6. The chapter is concluded by a list of more advanced topics (in Section 7) we have been forced not to consider in this brief survey.

1. Introduction

The perturbation theory of dynamical systems is called to explore the changes in dynamics as one perturbs (slightly modifies) the system at hand. Such studies are indispensable and of crucial importance for mathematics, natural science, and engineering due to two reasons.

First, while creating a mathematical model of a certain object or phenomenon in the real world, one usually neglects enormously many constituents that are not very relevant but can in principle affect the processes in question. For instance, the planetary system is modeled, as a rule, by a collection of several mass points — the star (Sun) and planets — interacting via Newton's law of gravitation. Here one does not take into account that the central star and all the planets have finite sizes, rotate, and can undergo tidal torques, that the system also includes such small bodies as moons of the planets, dwarf planets, asteroids, comets, and meteoroids, that the masses of the Sun and planets slowly change, etc. One also neglects cosmic dust, the solar radiation pressure, the gravitational attraction of various objects outside the given planetary system (such as other stars, black holes, and dark matter), the influence of dark energy, etc. Many of these effects are of non-Hamiltonian nature. Moreover, even if one confines oneself with examining the mathematical model itself, its parameters (the masses and initial positions and velocities of the star and planets in our example) can never be known with infinite precision. Consequently, only those features deserve study that are in a certain sense persistent under small perturbations of the model. A phenomenon that is characteristic of a particular system and completely disappears under arbitrarily small variations of the system cannot be expected to describe anything in the physical world.

Second, in many cases, mathematical models of real objects turn out to be close to extremely simple systems. For instance, the mutual gravitational attraction of the planets to each other is small compared to the attraction of the planets to the star (in our Solar System, the mass of the heaviest planet, Jupiter, is 0.0009546 times the mass of the Sun). Thus, "in the zero-th approximation", the motions of the planets can be represented as independent motions of non-interacting particles around a fixed attracting center along Keplerian ellipses. As a rule, the simple "unperturbed" systems one deals with are rather special (for example, they may possess additional symmetries absent in generic systems), and the actual behavior in the model at hand can be very far from the dynamics in its "unperturbed" version. However, the presence of "small parameters" in the model (the zero values of these parameters corresponding to the "unperturbed" case) usually facilitates the study of the model greatly. The methods for such studies in various situations and the results obtained constitute the core of the perturbation theory.

The present chapter is mainly devoted to nearly integrable Hamiltonian systems, i.e. Hamiltonian systems close to so-called completely integrable systems, see Section 3.1 below. Throughout the chapter, the notation $a \cdot b$ for vector quantities $a = (a_1, \dots, a_\ell)$ and $b = (b_1, \dots, b_\ell)$ means $a_1 b_1 + \cdots + a_\ell b_\ell$.

2. Simplest Persistence Problems

To fix thoughts, we start our exposition by some almost trivial persistence problems in Hamiltonian dynamics. The simplest dynamical pattern is an equilibrium point. Let a given autonomous Hamiltonian system $\dot{x} = V_0(x)$ with n degrees of freedom possess an equilibrium point O $(V(O) = 0)$. How does this affect the dynamics of nearby systems? The following proposition answers this question.

Theorem 2.1: *The $2n$ eigenvalues of the linearization of $\dot{x} = V_0(x)$ around O come in pairs $\lambda, -\lambda$. If all these eigenvalues are other than zero, then the equilibrium point O persists under small Hamiltonian perturbations of the system: any system $\dot{x} = V(x)$ with V sufficiently close to V_0 possesses an isolated equilibrium point close to O, and if V depends smoothly on some external parameters, so does the corresponding equilibrium point.*

This trivial result immediately follows from the implicit function theorem. In fact, the Hamiltonian nature of systems is irrelevant here (it manifests itself only in the symmetry of the spectra with respect to 0).

Now let us suppose that a Hamiltonian system $\dot{x} = V_0(x)$ with n degrees of freedom possesses a closed trajectory Γ. To this trajectory, one assigns the Poincaré return map ("first recurrence map") Π and the eigenvalues of the linearization of Π around the fixed point corresponding to Γ (the *multipliers* of Γ).

Theorem 2.2: *One of the multipliers of Γ is equal to* 1*, the other $2n-2$ multipliers coming in pairs λ, λ^{-1}. If all these $2n-2$ multipliers are other than* 1*, the trajectory Γ is included in a smooth one-parameter family of periodic trajectories with different periods (one trajectory per energy value). Moreover, this one-parameter family of closed trajectories persists under small Hamiltonian perturbations of the system (in the same sense as in the case of Theorem 2.1).*

This result is again an easy consequence of the implicit function theorem applied to the Poincaré section within an energy level hypersurface in the phase space.

Now observe that an equilibrium point is an invariant 0-torus $\mathbb{T}^0$, and a periodic trajectory is an invariant 1-torus $\mathbb{T}^1 = \mathbb{S}^1$ (a circle) without equilibrium points. On such a trajectory, one can introduce a uniformly rotating angular coordinate $\varphi \bmod 2\pi$ ($\dot{\varphi} \equiv \omega > 0$, where $2\pi/\omega$ is the period). As the next step, it is natural to consider an invariant 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ without equilibrium points and closed trajectories. It follows from some results by A. Denjoy (1932) and C. L. Siegel (1945) that any C^2 vector field on $\mathbb{T}^2$ without equilibria and periodic trajectories has a constant form (ω_1, ω_2) in a suitable coordinate frame $(\varphi_1, \varphi_2 \operatorname{modd} 2\pi)$, the frequency ratio ω_2/ω_1 being irrational (however, the transition functions from the original coordinates to the new ones cannot always be chosen to be smooth). The following concepts are central in the classical perturbation theory.

Definition 2.1: An invariant ℓ-torus $\mathcal{T} \approx \mathbb{T}^\ell = \left(\mathbb{S}^1\right)^\ell = \left(\mathbb{R}/2\pi\mathbb{Z}\right)^\ell$ of an autonomous (not necessarily Hamiltonian) flow is said to carry *conditionally periodic motions* with a frequency vector $\omega \in \mathbb{R}^\ell$ if the flow on $\mathcal{T}$ (sometimes called a *Kronecker flow* in this case) takes the constant form $\dot{\varphi} \equiv \omega$ in a suitable coordinate frame $\left(\varphi_1, \ldots, \varphi_\ell \operatorname{modd} 2\pi\right)$. This torus is said to be *non-resonant* if the frequencies $\omega_1, \ldots, \omega_\ell$ are incommensurable (linearly independent over rationals) and is said to be *resonant* otherwise. Conditionally periodic motions with incommensurable frequencies are called *quasi-periodic motions*.

Any trajectory on an invariant ℓ-torus $\mathcal{T}$ carrying quasi-periodic motions is dense in $\mathcal{T}$. On the other hand, if the frequencies $\omega_1, \ldots, \omega_\ell$ of conditionally periodic motions on $\mathcal{T}$ satisfy exactly r independent resonance relations $k^{(\iota)} \cdot \omega = 0$ with $k^{(\iota)} \in \mathbb{Z}^\ell \setminus \{0\}$, $\iota = 1, \ldots, r$ ($1 \le r \le \ell$, the number r is called the *resonance multiplicity*), then $\mathcal{T}$ is foliated into invariant $(\ell - r)$-tori carrying quasi-periodic motions with the same frequency vector $\hat{\omega} \in \mathbb{R}^{\ell - r}$. Resonant invariant tori carrying conditionally periodic motions are therefore highly "degenerate" and impossible in generic (Hamiltonian as well as non-Hamiltonian) systems.

Being inspired by Theorems 2.1 and 2.2 above, one could expect to encounter continuous two-parameter families of invariant 2-tori in Hamiltonian systems, each torus carrying conditionally periodic motions with some frequencies $\omega_1(\mu)$ and $\omega_2(\mu)$ where μ denotes the two-dimensional parameter of the family.

Nevertheless, it is not hard to realize that such families cannot exist as persistent structures. Indeed, if the frequencies $\omega_1(\mu)$ and $\omega_2(\mu)$ depend continuously on μ, then, generally speaking, they become commensurable for some μ (i.e., the corresponding 2-torus becomes resonant), and such values of μ constitute a dense set. But generic systems do not admit invariant 2-tori foliated into closed trajectories with the same period. Invariant ℓ-tori of dimensions $\ell \ge 2$ turn out to be much "subtler" than equilibria and closed trajectories.

Notwithstanding, one of the most important and astonishing discoveries of 20$^{\text{th}}$ century mathematics is that Hamiltonian systems with n degrees of freedom *do* possess persistent ℓ-parameter families of invariant ℓ-tori carrying quasi-periodic motions for each $\ell = 2, \ldots, n$. But these families are not continuous; they are Cantor-like (the parameter labeling the tori ranges in a nowhere dense subset of $\mathbb{R}^\ell$ of positive Lebesgue measure). Precise statements describing such families will be given below in Section 5.1.

The ubiquity of invariant tori carrying conditionally periodic motions in dynamical systems stems, in the long run, from the fact that any finite-dimensional connected compact Abelian Lie group is a torus (where the group operation is the addition).

Throughout this chapter, all the invariant tori in Hamiltonian systems we deal with carry conditionally periodic motions. It is worthwhile to emphasize, however, that a Hamiltonian system can possess an invariant ℓ-torus with *any* prescribed dynamics $\dot{\varphi} = f(\varphi)$, $\varphi \in \mathbb{T}^\ell$. Indeed, consider a Hamiltonian system with ℓ degrees of freedom, phase space variables $I \in \mathbb{R}^\ell$, $\varphi \in \mathbb{T}^\ell$ (I ranging near the origin), the symplectic 2-form $dI \wedge d\varphi = dI_1 \wedge d\varphi_1 + \cdots + dI_\ell \wedge d\varphi_\ell$, and the Hamilton function $I_1 f_1(\varphi) + \cdots + I_\ell f_\ell(\varphi)$. The corresponding equations of motion are

$$\dot{I}_i = -I_1 \frac{\partial f_1(\varphi)}{\partial \varphi_i} - \cdots - I_\ell \frac{\partial f_\ell(\varphi)}{\partial \varphi_i}, \qquad \dot{\varphi}_i = f_i(\varphi),$$

$i = 1, \ldots, \ell$. The ℓ-torus $\{I = 0\}$ is invariant, the induced flow being given by the equation $\dot{\varphi} = f(\varphi)$. Nonetheless, it is invariant tori carrying quasi-periodic motions that are *typical* in Hamiltonian systems.

3. Integrable and Partially Integrable Systems

First we consider exceptional Hamiltonian systems which admit smooth families of invariant tori carrying conditionally periodic motions. Such families occur in Hamiltonian systems exhibiting additional (besides the autonomous Hamilton function itself) first integrals (in the whole phase space or within a certain invariant surface).

Recall that a submanifold M of a $2n$-dimensional symplectic manifold $\mathfrak{W}^{2n}$ is said to be *isotropic* if the restriction of the symplectic 2-form to M vanishes (and, consequently, $\dim M \le n$). A submanifold $M \subset \mathfrak{W}^{2n}$ is said to be *Lagrangian* if it is isotropic and its dimension is equal to the maximal possible value n. Two functions on $\mathfrak{W}^{2n}$ are said to be *in involution* if their Poisson bracket vanishes. If functions $F_1, \ldots, F_m$ on $\mathfrak{W}^{2n}$ are independent and pairwise in involution then $m \le n$.

According to M. R. Herman's theorem (1988), an invariant torus of a Hamiltonian system is necessarily isotropic (and, consequently, its dimension does not exceed the number of degrees of freedom) provided that it carries quasi-periodic motions and the symplectic 2-form is *exact*.

3.1. Action-Angle Variables. Liouville–Arnold Theorem

Hamiltonian systems with n degrees of freedom that have n independent integrals in involution are described by the following fundamental theorem.

Liouville–Arnold theorem: *Suppose that an autonomous Hamiltonian system with n degrees of freedom and Hamilton function H possesses n smooth integrals*

$F_1 = H, F_2, \ldots, F_n$ *that are pairwise in involution. Let* M *be a connected component of one of the common level surfaces* $\{F_i = c_i, 1 \le i \le n\}$ *of these integrals, and let the differentials of the functions* $F_1, \ldots, F_n$ *be linearly independent at each point of the set* M *(in other words, let the rank of the Jacobi matrix of these functions be equal to* n *everywhere on* M *). Then* M *is a smooth Lagrangian submanifold of the phase space. Moreover, assume that whenever a trajectory of any of the Hamiltonian systems with Hamilton functions* $F_1, \ldots, F_n$ *lies on* M *, it is defined for all* $t \in \mathbb{R}$ *. Then:*

a) *The surface* M *is diffeomorphic to the product* $\mathbb{T}^{\mathfrak{s}} \times \mathbb{R}^{n-\mathfrak{s}}$ *of the* $\mathfrak{s}$*-torus* $\mathbb{T}^{\mathfrak{s}}$ *and the* $(n-\mathfrak{s})$*-dimensional Euclidean space* $\mathbb{R}^{n-\mathfrak{s}}$ *for a certain* $\mathfrak{s}$ *in the range* $0 \le \mathfrak{s} \le n$.

b) *In* $\mathbb{T}^{\mathfrak{s}} \times \mathbb{R}^{n-\mathfrak{s}}$, *one can introduce coordinates* $\varphi = (\varphi_1, \ldots, \varphi_{\mathfrak{s}}) \in \mathbb{T}^{\mathfrak{s}}$, $x = (x_1, \ldots, x_{n-\mathfrak{s}}) \in \mathbb{R}^{n-\mathfrak{s}}$ *in which the Hamilton equations with Hamilton functions* F_i $(1 \le i \le n)$ *on* M *take the form*

$$\dot{\varphi}_j = \omega_{ji}, \qquad \dot{x}_l = a_{li}$$

with constant ω_{ji}, a_{li} $(1 \le j \le \mathfrak{s}, 1 \le l \le n-\mathfrak{s})$.

c) *The Hamilton equations with Hamilton functions* $F_1, \ldots, F_n$ *can be integrated by quadratures.*

d) *Suppose additionally that the manifold* M *is compact, i.e.,* $\mathfrak{s} = n$ *(in this case, the condition of infinite extendibility of the trajectories on* M *is fulfilled automatically). Then some small neighborhood of the surface* M *in the phase space is diffeomorphic to the product* $D \times \mathbb{T}^n$ *of an open domain* D *in the* n*-dimensional Euclidean space* $\mathbb{R}^n$ *and the n-torus* $\mathbb{T}^n$ *and, moreover, there are coordinates* $I = (I_1, \ldots, I_n) \in D$, $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{T}^n$ *in* $D \times \mathbb{T}^n$ *with the following properties:*

   i) *the torus* M *is given by the equation* $I = I^*$ *for a certain* $I^* \in D$*;*
   ii) *the functions* $F_1, \ldots, F_n$ *in the variables* (I, φ) *depend on* I *only;*
   iii) *the symplectic* 2*-form is* $dI \wedge d\varphi = dI_1 \wedge d\varphi_1 + \cdots + dI_n \wedge d\varphi_n$.

In particular, properties ii) *and* iii) *imply that in the coordinates* (I, φ), *the Hamilton equations with Hamilton functions* F_i $(1 \le i \le n)$ *in a neighborhood of the manifold* M *have the form*

$$\dot{I} = 0, \quad \dot{\varphi} = \partial F_i(I)/\partial I.$$

The name of this theorem reflects landmark contributions by J. Liouville (1855) and V. I. Arnold (1963). In the literature, the theorem is also referred to as the action-angle theorem, Arnold–Liouville theorem, Arnold theorem, Arnold–Jost theorem, Liouville–Arnold–Jost theorem, or Liouville–Arnold–Mineur theorem (with any order of the people involved). These names remind one of contributions by H. Mineur (1936) and R. Jost (1968).

A Hamiltonian system with Hamilton function H satisfying the hypotheses of this theorem is said to be *completely integrable* or *Liouville integrable* (in a neighborhood of the manifold M). The coordinates (I,φ) one speaks of in item d) of the theorem are called the *action-angle variables*. Under the hypotheses of item d), a neighborhood of the manifold M is foliated into Lagrangian invariant n-tori $\{I=\mathrm{const}\}$ (called *Liouville tori*) of the completely integrable system (M is one of these tori), the motions on the tori being conditionally periodic with frequency vectors $\omega(I)=\partial H(I)/\partial I$.

For instance, a Hamiltonian system with one degree of freedom is Liouville integrable in any domain where there are no equilibrium points and each trajectory is infinitely extendible. If a planar Hamiltonian system possesses a family of closed trajectories Γ, then the actions corresponding to these trajectories are equal (up to an arbitrary additive constant) to $\mathrm{Area}(\Gamma)/2\pi$ where $\mathrm{Area}(\Gamma)$ are the areas enclosed by the trajectories Γ (the area element being given by the symplectic 2-form).

While speaking of completely integrable Hamiltonian systems, one almost always has in view the case of compact common level surfaces of the integrals, i.e., item d) of the Liouville–Arnold theorem. In the present chapter, we will observe this rule.

3.2. Partially Integrable Systems

Some Hamiltonian systems with n degrees of freedom encountered in practice admit smooth 2ℓ-dimensional invariant surfaces foliated into isotropic invariant ℓ-tori carrying conditionally periodic motions, $2\le\ell\le n-1$. Such systems are sometimes said to be *partially integrable*. For instance, consider an autonomous Hamiltonian system with $\ell+m$ degrees of freedom and with the Hamilton function of the form

$$H(I,\varphi,z)=F(I)+\tfrac{1}{2}Q(I,\varphi)z\cdot z+R(I,\varphi,z), \qquad (1)$$

where

a) (I,φ,z) are the phase space variables, $I=(I_1,\dots,I_\ell)$ ranges in an open domain $D\subset\mathbb{R}^\ell$, $\varphi=(\varphi_1,\dots,\varphi_\ell)$ ranges in $\mathbb{T}^\ell$, and $z=(z_1,\dots,z_{2m})$ ranges near the origin of $\mathbb{R}^{2m}$;

b) the symplectic 2-form is

$$\sum_{i=1}^{\ell}dI_i\wedge d\varphi_i+\sum_{s=1}^{m}dz_s\wedge dz_{s+m};$$

c) $Q(I,\varphi)$ is a symmetric $2m\times 2m$ matrix depending on I and φ while $R=\mathcal{O}\left(|z|^3\right)$, this notation means that the Taylor expansion of the remainder R in z starts with terms of order 3.

The Hamilton function H affords the equations of motion

$$\dot{I} = \mathcal{O}\left(|z|^2\right), \quad \dot{\varphi} = \omega(I) + \mathcal{O}\left(|z|^2\right), \quad \dot{z} = \Omega(I,\varphi)z + \mathcal{O}\left(|z|^2\right), \qquad (2)$$

where

$$\omega(I) = \frac{\partial F(I)}{\partial I}, \quad \Omega(I,\varphi) = \mathcal{J}Q(I,\varphi), \quad \mathcal{J} = \begin{pmatrix} 0_m & -E_m \\ E_m & 0_m \end{pmatrix} \qquad (3)$$

(0_m and E_m being the zero and identity $m \times m$ matrices, respectively).

One sees that the 2ℓ-dimensional surface $\{z = 0\}$ is invariant and foliated into isotropic invariant ℓ-tori $\{z = 0, I = \text{const}\}$ carrying conditionally periodic motions with frequency vectors $\omega(I)$. The restriction of the system (2) to this surface is completely integrable with action-angle variables (I,φ). Recall the following important concepts.

Definition 3.2.1: Let an invariant ℓ-torus $\mathcal{T}$ of an autonomous (not necessarily Hamiltonian) flow on an $(\ell + N)$-dimensional manifold carry conditionally periodic motions with frequency vector $\omega \in \mathbb{R}^\ell$. This torus is said to be *reducible* (or *Floquet*) if in a neighborhood of $\mathcal{T}$, there exists a coordinate frame $\varphi \in \mathbb{T}^\ell$, $X \in \mathbb{R}^N$ (X ranging near the origin) in which the torus $\mathcal{T}$ itself is given by the equation $X = 0$ and the dynamical system takes the *Floquet form* $\dot{\varphi} = \omega + \mathcal{O}(|X|)$, $\dot{X} = LX + \mathcal{O}\left(|X|^2\right)$ with a φ-independent $N \times N$ matrix L. This matrix is called the *Floquet matrix* of the torus $\mathcal{T}$, and its eigenvalues are called the *Floquet exponents* of $\mathcal{T}$.

In other words, an invariant torus is reducible if the variational equations along this torus can be reduced to a form with constant coefficients. For instance, the invariant tori $\{I = \text{const}\}$ of a completely integrable Hamiltonian system are reducible with zero Floquet matrix.

If the matrix Q in the Hamilton function (1) does not depend on the angles φ, then the invariant tori $\{z = 0, I = \text{const}\}$ are reducible. Their $(\ell + 2m) \times (\ell + 2m)$ Floquet matrices are block diagonal with blocks 0_ℓ and $\Omega(I) = \mathcal{J}Q(I)$. The corresponding Floquet exponents are $\underbrace{0,\ldots,0}_{\ell}$ and the eigenvalues of $\Omega(I)$, these eigenvalues coming in pairs $\lambda, -\lambda$.

An isotropic invariant torus $\mathcal{T}$ of a Hamiltonian system is said to be *lower dimensional* if $\mathcal{T}$ carries conditionally periodic motions and its dimension is less than the number of degrees of freedom. So, partially integrable systems possess smooth families of lower dimensional invariant tori, the number of parameters of the family being equal to the dimension of the tori. By the way, Lagrangian invariant tori carrying conditionally periodic motions are sometimes said to be *maximal*.

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Biographical Sketch

Mikhail B. Sevryuk was born in Moscow in 1962. He graduated from the Moscow State University in 1984. In 1984–1987, he was a post-graduate student there (under the supervision of Professor V. I. Arnold). He defended his Ph.D. Thesis at this university in 1988. Since 1987, he has been working at the Institute of Energy Problems of Chemical Physics (Moscow), the USSR (Russia since 1991) Academy of Sciences, where he defended his Habilitation Thesis in 2003. His research activities belong to two almost disjoint areas of science: first, KAM theory and the theory of so-called reversible dynamical systems; second, the theory of elementary processes in Chemical Physics. He has participated in several international research projects in both Mathematics and Chemistry. He is the author/a coauthor of two monographs on KAM theory and related topics (published by Springer-Verlag in the series "Lecture Notes in Mathematics"), of about 100 papers in scientific journals, of several book chapters, and of a problem book for high school students interested in mathematics (published in Moscow).