ALGEBRAIC GEOMETRY AND APPLICATIONS

Tadao ODA
Tohoku University, Japan

Keywords: algebraic set, maximal ideal space, Hilbert's Nullstellensatz, Noether's normalization theorem, affine, projective, separated, algebraic variety, sheaf, scheme, product, local ring, stalk, local homomorphism, morphism, Zariski tangent space, singular, nonsingular, Jacobian matrix, complex analytic space, resolution, alteration, valuation ring, discrete valuation ring, Mori's Minimal Model Program, function field, complete, proper, normal, birational, divisor, complete linear system, Riemann-Roch theorem, algebraic curve, algebraic surface, error-correcting code

Contents

Summary

Algebraic geometry deals with geometric objects defined algebraically. The set of solutions (in complex numbers) of a system of algebraic equations, called an affine algebraic set, is first given an intrinsic formulation as the maximal ideal space of a finitely generated algebra over complex numbers. This formulation and the notion of sheaves enable one to glue affine algebraic sets together and obtain algebraic sets. Interesting among them are algebraic varieties whose basic geometric properties are explained. More general notions of schemes and algebraic varieties over arbitrary fields are introduced. As one of the applications, algebro-geometric coding system is sketched.

Due to space limitation, practically no examples are given to illustrate the rich geometry behind the abstract formulation. The readers are referred to the references listed at the end for illuminating self-contained description as well as for good references at more advanced level.

Results on matrices and linear algebra in *Matrices, Vectors, Determinants and Linear Algebra*, on groups in *Groups and Applications*, on rings and modules in *Rings and Modules* and on fields and algebraic equations in *Fields and Algebraic Equations* will be freely used.

1. Affine Algebraic Varieties

A very basic geometric object defined algebraically is the set of solutions of a system of algebraic equations.

To fix the ideas, let us consider the n-dimensional complex affine space

$$\mathbb{C}^n := \{(a_1, a_2, \ldots, a_n) \mid a_j \in \mathbb{C},\ j = 1, 2, \ldots, n\}.$$

Given polynomials

$$f_1(t_1, t_2, \ldots, t_n),\ f_2(t_1, t_2, \ldots, t_n), \ldots,\ f_l(t_1, t_2, \ldots, t_n)$$

in n variables $t_1, t_2, \ldots, t_n$ with coefficients in the field $\mathbb{C}$ of complex numbers, the system of algebraic equations

$$\begin{cases} f_1(t_1, t_2, \ldots, t_n) & = & 0 \\ f_2(t_1, t_2, \ldots, t_n) & = & 0 \\ \quad\vdots & & \\ f_l(t_1, t_2, \ldots, t_n) & = & 0 \end{cases}$$

gives rise to the set $X \subset \mathbb{C}^n$ of its solutions

$$X := \{a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n \mid f_1(a) = 0,\ f_2(a) = 0, \ldots,\ f_l(a) = 0\},$$

where we use the simplified notation $f_j(a) := f_j(a_1, a_2, \ldots, a_n)$ for $j = 1, 2, \ldots, l$.

Subsets of $\mathbb{C}^n$ of this form are called *affine algebraic subsets*.

Consider the polynomial ring $\mathbb{C}[t_1, t_2, \ldots, t_n]$ in n variables $t_1, t_2, \ldots, t_n$ with complex coefficients. The notation $\mathbb{C}[t] := \mathbb{C}[t_1, t_2, \ldots, t_n]$ will be used for simplicity when there is no fear of confusion.

In the description of an algebraic set X above, consider the ideal

$$\mathcal{I} := \left\{\varphi_1 f_1 + \varphi_2 f_2 + \cdots + \varphi_l f_l \,\middle|\, \varphi_1, \varphi_2, \ldots, \varphi_l \in \mathbb{C}[t]\right\}.$$

of $\mathbb{C}[t]$ generated by $f_1, f_2, \ldots, f_l$. Obviously, one has

$$X = \{a \in \mathbb{C}^n \mid f(a) = 0,\ \forall f \in \mathcal{I}\}.$$

By *Hilbert's basis theorem*, any ideal $\mathcal{J}$ of $\mathbb{C}[t]=\mathbb{C}[t_1,\ldots,t_n]$ is necessarily finitely generated, that is, there exist $g_1, g_2,\ldots, g_s \in \mathbb{C}[t]$ such that

$$\mathcal{J}=\left\{\psi_1 g_1+\psi_2 g_2+\cdots+\psi_s g_s \,\middle|\, \psi_1,\ldots,\psi_s \in \mathbb{C}[t]\right\}.$$

Thus one could have defined algebraic subsets of $\mathbb{C}^n$ to be those of the form

$$X_{\mathcal{I}} := \{a=(a_1,\ldots,a_n)\in\mathbb{C}^n \mid f(a)=0,\ \forall f\in\mathcal{I}\}$$

for an ideal $\mathcal{I}\subset\mathbb{C}[t]$.

For two ideals $\mathcal{I}$ and $\mathcal{J}$ of $\mathbb{C}[t]$ with $\mathcal{I}\subset\mathcal{J}$, one obviously has $X_{\mathcal{I}}\supset X_{\mathcal{J}}$.
The *radical* $\sqrt{\mathcal{I}}$ of an ideal $\mathcal{I}\subset\mathbb{C}[t]$ is defined to be

$$\sqrt{\mathcal{I}} := \left\{g\in\mathbb{C}[t] \,\middle|\, g^r\in\mathcal{I} \text{ for a positive integer } r\right\}.$$

An important theorem known as *Hilbert's Nullstellensatz* says that for ideals $\mathcal{I}$ and $\mathcal{J}$ of $\mathbb{C}[t]$ one has

$$X_{\mathcal{I}} = X_{\mathcal{J}} \Longleftrightarrow \sqrt{\mathcal{I}}=\sqrt{\mathcal{J}}.$$

To motivate further considerations of algebraic subsets, consider, for instance,

$$X := \{(a_1,a_2) \mid a_1^2+a_2^2=1\}\subset\mathbb{C}^2,$$

which is an important geometric object with its "real locus" $X\cap\mathbb{R}^2$ equal to the circle of radius one centered at the origin. The change of coordinates

$$\begin{cases} u_1 &= t_1+\sqrt{-1}t_2 \\ u_2 &= t_1-\sqrt{-1}t_2 \end{cases}$$

in $\mathbb{C}^2$ turns X into an algebraic subset of different shape

$$Y := \{(b_1,b_2)\in\mathbb{C}^2 \mid b_1 b_2 = 1\},$$

since $t_1^2+t_2^2=u_1u_2$. Although the real loci $X\cap\mathbb{R}^2$ and $Y\cap\mathbb{R}^2$ have completely different geometric properties, one would like to regard X and Y to be intrinsically the same at least as geometric objects in $\mathbb{C}^n$.

For that purpose, regard an element $f(t)=f(t_1,t_2,\ldots,t_n)\in\mathbb{C}[t]$ as a "polynomial function" on $\mathbb{C}^n$ defined by

$$\mathbb{C}^n \ni a=(a_1,a_2,\ldots,a_n)\mapsto f(a)\in\mathbb{C}.$$

Given an algebraic subset $X_{\mathcal{I}}$ for an ideal $\mathcal{I}\subset\mathbb{C}[t]$, the restrictions to $X_{\mathcal{I}}$ of polynomial functions $g(t),h(t)\in\mathbb{C}[t]$ give rise to the same function if the difference $g(t)-h(t)$ belongs to the ideal $\mathcal{I}$. Thus the residue class ring

$$A(X_{\mathcal{I}}):=\mathbb{C}[t]/\mathcal{I}$$

can be regarded as the set of "polynomial functions" on the algebraic subset $X_{\mathcal{I}}$.

Here are two important observations which enable one to recover $X_{\mathcal{I}}$ from the ring $A(X_{\mathcal{I}})$ of its polynomial functions:

- Through the inclusion map $\mathbb{C}\to\mathbb{C}[t]$ sending $c\in\mathbb{C}$ to the constant polynomial c, one regards $\mathbb{C}[t]$ as a $\mathbb{C}$-algebra. A $\mathbb{C}$-algebra homomorphism $\alpha:\mathbb{C}[t]\to\mathbb{C}$ (i.e., a ring homomorphism whose restriction to $\mathbb{C}\subset\mathbb{C}[t]$ is the identity map to $\mathbb{C}$) is uniquely determined by $\alpha(t_1),\alpha(t_2),\ldots,\alpha(t_n)\in\mathbb{C}$, and one has a bijective correspondence

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[t_1,t_2,\ldots,t_n],\mathbb{C})\xrightarrow{\sim}\mathbb{C}^n$$

sending a $\mathbb{C}$-algebra homomorphism α to $(\alpha(t_1),\alpha(t_2),\cdots,\alpha(t_n))\in\mathbb{C}^n$. Here and elsewhere, $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(B,B')$ denotes the set of $\mathbb{C}$-algebra homomorphisms from a $\mathbb{C}$-algebra B to another $\mathbb{C}$-algebra B'.

For the residue class ring $A:=A(X_{\mathcal{I}})=\mathbb{C}[t]/\mathcal{I}$, the set $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(A,\mathbb{C})$ of $\mathbb{C}$-algebra homomorphisms can be identified with the subset of $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[t],\mathbb{C})$ consisting of those α 's such that $\alpha(\mathcal{I})=0$. Thus one has a bijective correspondence

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[t],\mathbb{C})\supset\mathrm{Hom}_{\mathbb{C}\text{-alg}}(A(X_{\mathcal{I}}),\mathbb{C})\xrightarrow{\sim}X_{\mathcal{I}}\subset\mathbb{C}^n.$$

- The kernel of a $\mathbb{C}$-algebra homomorphism $\alpha:\mathbb{C}[t]\to\mathbb{C}$ is obviously a maximal ideal of $\mathbb{C}[t]$. As a consequence of an important theorem known as *Noether's normalization theorem* together with the fact that $\mathbb{C}$ is algebraically closed (*the fundamental theorem of algebra*), one sees that all maximal ideals of $\mathbb{C}[t]$ arise in this way, and one has a bijective correspondence

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[t],\mathbb{C})\xrightarrow{\sim}\mathrm{Max}(\mathbb{C}[t]),\quad \alpha\mapsto\ker(\alpha),$$

where $\mathrm{Max}(\mathbb{C}[t])$ denotes the *set of maximal ideals* of $\mathbb{C}[t]$. A maximal ideal of $A = A(X_{\mathcal{I}}) = \mathbb{C}[t]/\mathcal{I}$ is of the form $\mathcal{M}/\mathcal{I}$ for a maximal ideal $\mathcal{M}$ of $\mathbb{C}[t]$ satisfying $\mathcal{M} \supset \mathcal{I}$. Hence the set $\mathrm{Max}(A)$ of maximal ideals of A may be regarded as a subset of $\mathrm{Max}(\mathbb{C}[t])$ consisting of those maximal ideals $\mathcal{M}$ satisfying $\mathcal{M} \supset \mathcal{I}$. Thus one has a bijective correspondence

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[t], \mathbb{C}) \supset \mathrm{Hom}_{\mathbb{C}\text{-alg}}(A(X_{\mathcal{I}}), \mathbb{C}) \xrightarrow{\sim} \mathrm{Max}(A(X_{\mathcal{I}})) \subset \mathrm{Max}(\mathbb{C}[t]).$$

In terms of the $\mathbb{C}$-algebra $A(X_{\mathcal{I}}) := \mathbb{C}[t]/\mathcal{I}$, one thus has bijections

$$X_{\mathcal{I}} \xleftarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(A(X_{\mathcal{I}}), \mathbb{C}) \xrightarrow{\sim} \mathrm{Max}(A(X_{\mathcal{I}})).$$

$X_{\mathcal{I}}$ is given originally as a subset of $\mathbb{C}^n$. This information can be recovered from $A(X_{\mathcal{I}})$ as the choice of a system of $\mathbb{C}$-algebra generators $\overline{t_1}, \ldots, \overline{t_n} \in A(X_{\mathcal{I}})$, which are the images of $t_1, \ldots, t_n$ under the canonical surjection $\mathbb{C}[t] \to A(X_{\mathcal{I}})$.

The *nilradical* $\sqrt{\{0\}}$ of the ring $A = \mathbb{C}[t]/\mathcal{I}$, which is the ideal consisting of nilpotent elements of A, is nothing but $\sqrt{\mathcal{I}}/\mathcal{I}$. One denotes

$$A_{\mathrm{red}} := A/\sqrt{\{0\}} = \mathbb{C}[t]/\sqrt{\mathcal{I}}.$$

As a consequence of Hilbert's Nullstellensatz, one thus has $\mathrm{Max}(A) = \mathrm{Max}\left(A_{\mathrm{red}}\right)$.

In conclusion, the very basic among algebraically defined geometric objects of interest can now be intrinsically defined as the sets

$$\mathrm{Max}(A) = \mathrm{Max}\left(A_{\mathrm{red}}\right) \quad \text{for finitely generated } \mathbb{C}\text{-algebras } A,$$

which are called *affine algebraic sets*. (Finitely generated $\mathbb{C}$-algebras are also called $\mathbb{C}$-algebras of *finite type*.)

An affine algebraic set $X_A := \mathrm{Max}(A)$ is not just a point set but has rich geometry hidden inside the $\mathbb{C}$-algebra A.

Let us denote by $\mathfrak{m}(x) \in \mathrm{Max}(A)$ the maximal ideal corresponding to each $x \in X_A$. First of all, it is a topological space with the *Zariski topology* in which the *closed subsets* of X are defined as those subsets of the form

$$X_A(I) := \{x \in X_A \mid \mathfrak{m}(x) \supset I\} \quad \text{for ideals } I \subset A.$$

One sees that $X_A(I) = \mathrm{Max}(A/I)$, hence closed subsets of X_A are affine algebraic sets as well.

By definition, the *open subsets* of X_A are those subsets of the form

$$U(I) := X_A \setminus X_A(I) = \{x \in X_A \,|\, \mathfrak{m}(x) \not\supset I\} = \bigcup_{f \in I} U_f, \quad \text{with } U_f := \{x \in X_A \,|\, \mathfrak{m}(x) \not\ni f\}.$$

Open sets of the form U_f for $f \in A$ are called *distinguished open sets*, and they thus form an *open basis* for the Zariski topology of X_A.

$U_f = \emptyset$ if f is a nilpotent element. Otherwise, consider the ring of quotients $A[1/f]$ of A with respect to the multiplicatively closed set $S := \{1, f, f^2, f^3, \ldots\}$. Since maximal ideals $\mathfrak{m}(x)$ with $f \notin \mathfrak{m}(x)$ are in one-to-one correspondence with the maximal ideals of $A[1/f]$, one has a natural identification

$$\mathrm{Max}(A) \supset U_f = \mathrm{Max}(A[1/f]).$$

If X_A is defined as before as an algebraic subset of $\mathbb{C}^n$ by $A = \mathbf{C}[t_1,\ldots,t_n]/\mathcal{I}$, then X_A is the closed subset of $\mathbb{C}^n = \mathrm{Max}(\mathbb{C}[t_1,t_2,\ldots,t_n])$ defined by the ideal $\mathcal{I}$ with respect to the Zariski topology of the latter. The Zariski topology of X_A is then the topology induced by the Zariski topology of $\mathbb{C}^n$.

Given a $\mathbb{C}$-algebra homomorphism $\varphi : B \to A$ from a $\mathbb{C}$-algebra B to another A, one has a natural map

$$\varphi^* : X_A \to X_B \quad \text{with} \quad \mathfrak{m}(\varphi^*(x)) := \varphi^{-1}(\mathfrak{m}(x)) \quad \text{for } x \in X_A,$$

which is called a *morphism* corresponding to φ and can be easily seen to be *continuous* with respect to the Zariski topologies of X_A and X_B.

When X_A and X_B are given as closed algebraic subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, by $A = \mathbb{C}[t_1,\ldots,t_n]/\mathcal{I}$ and $B = \mathbb{C}[u_1,\ldots,u_m]/\mathcal{J}$ with ideals satisfying $\sqrt{\mathcal{I}} = \mathcal{I}$ and $\sqrt{\mathcal{J}} = \mathcal{J}$, then $\mathbb{C}$-algebra homomorphisms $\varphi : B \to A$ are in one-to-one correspondence with the $\mathbb{C}$-algebra homomorphisms $\Phi : \mathbb{C}[u_1,\ldots,u_m] \to \mathbb{C}[t_1,\ldots,t_n]$ such that $\Phi(\mathcal{J}) \subset \mathcal{I}$. A $\mathbb{C}$-algebra homomorphism $\Phi : \mathbb{C}[u_1,\ldots,u_m] \to \mathbb{C}[t_1,\ldots,t_n]$ is equivalent to giving polynomials

$$h_1 := \Phi(u_1), h_2 := \Phi(u_2), \ldots, h_m := \Phi(u_m) \in \mathbb{C}[t_1,t_2,\ldots,t_n],$$

which give a "polynomial map"

$$\Phi^* : \mathbb{C}^n \longrightarrow \mathbb{C}^m, \quad \text{with } \mathbb{C}^n \ni a = (a_1, a_2, \ldots, a_n) \mapsto (h_1(a), h_2(a), \ldots, h_m(a)) \in \mathbb{C}^m.$$

Thus a morphism $\varphi^* : X_A \to X_B$ is exactly a map induced by a "polynomial map" $\Phi : \mathbb{C}^n \to \mathbb{C}^m$ satisfying $\Phi(X_A) \subset X_B$.

Given finitely generated $\mathbb{C}$-algebras A and A', the product set $X_A \times X_{A'}$ has a natural structure of an affine algebraic set by

$$X_A \times X_{A'} = X_{A \otimes_{\mathbb{C}} A'},$$

since one easily sees that

$$\begin{aligned}&\mathrm{Max}(A \otimes_{\mathbb{C}} A') = \mathrm{Hom}_{\mathbb{C}\text{-alg}}(A \otimes_{\mathbb{C}} A', \mathbb{C}) \\ &= \mathrm{Hom}_{\mathbb{C}\text{-alg}}(A, C) \times \mathrm{Hom}_{\mathbb{C}\text{-alg}}(A', \mathbb{C}) = \mathrm{Max}(A) \times \mathrm{Max}(A'),\end{aligned}$$

where $A \otimes_{\mathbb{C}} A'$ is the tensor product of $\mathbb{C}$-algebras, that is, the tensor product of A and A' as $\mathbb{C}$-vector spaces together with the multiplication defined by $(a \otimes a')(b \otimes b') := ab \otimes a'b'$ for $a, b \in A$ and $a', b' \in A'$.

Except in special cases, the Zariski topology of this affine algebraic set is much stronger than the product of the Zariski topologies of X_A and $X_{A'}$.

On the other hand, the Zariski topology of X_A does not satisfy the *Hausdorff axiom* of point separation except in special cases. Instead, the *diagonal morphism*

$$\Delta : X_A \longrightarrow X_A \times X_A = X_{A \otimes_{\mathbb{C}} A}, \quad \text{with } \Delta(x) := (x, x) \text{ for } x \in X_A,$$

which corresponds to the $\mathbb{C}$-algebra multiplication homomorphism

$$\mu : A \otimes_{\mathbb{C}} A \to A, \quad \mu(a \otimes a') := aa',$$

identifies X_A with the Zariski closed subset of $X_A \times X_A$ defined by the "diagonal ideal" $\ker(\mu) \subset A \otimes_{\mathbb{C}} A$. Note that

$$f(aa') = f(a) f(a') = (f \otimes f)(a \otimes a'), \quad \forall f \in \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C}), \quad \forall a, a' \in A.$$

This property of the diagonal map Δ being a "closed immersion" is called the *separatedness* of the affine algebraic set X_A.

An affine algebraic set X_A is said to be an *affine algebraic variety* if the $\mathbb{C}$-algebra A is an *integral domain*, that is, A has no zero divisors other than 0. (To be more precise, an affine algebraic variety here is called an affine algebraic variety *over* $\mathbb{C}$ or a *complex* affine algebraic variety.) Any affine algebraic set X_A turns out to be expressed uniquely as a finite union

$$X_A = V_1 \cup V_2 \cup \cdots \cup V_r, \quad \text{with } V_i \not\subset V_j \text{ whenever } i \neq j$$

of closed subsets $V_1, V_2, \ldots, V_r$ that are affine algebraic varieties called the *irreducible components* of X_A. Each of these irreducible components cannot be expressed further as a finite union of closed subsets in a nontrivial way. This fact corresponds to the fact that the nilradical $\sqrt{\{0\}}$ of A is the intersection of *minimal* prime ideals of A.

2. Projective Algebraic Varieties

Affine algebraic varieties are too restrictive and need to be generalized for truly interesting geometry. Before introducing general algebraic varieties later, let us introduce projective algebraic varieties.

The n-dimensional *complex projective space* is defined as the quotient space

$$\mathbb{P}^n(\mathbb{C}) := \left(\mathbb{C}^{n+1} \setminus \{O\}\right)/\mathbb{C}^\times,$$

where $\mathbb{C}^{n+1} \setminus \{O\}$ is the $(n+1)$-dimensional complex affine space minus the origin $O := (0, 0, \ldots, 0)$, with coordinates $t = (t_0, t_1, \ldots, t_n)$, on which the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers acts by

$$\lambda(t_0, t_1, \ldots, t_n) := (\lambda t_0, \lambda t_1, \ldots, \lambda t_n), \quad \lambda \in \mathbb{C}^\times, \quad (t_0, t_1, \ldots, t_n) \in \mathbb{C}^{n+1} \setminus \{O\}.$$

A point of $\mathbb{P}^n(\mathbb{C})$ is thus a $\mathbb{C}^\times$-orbit

$$[t_0, t_1, \ldots, t_n] := \{(\lambda t_0, \lambda t_1, \ldots, \lambda t_n) \mid \lambda \in \mathbb{C}^\times\}$$

of a point $(t_0, t_1, \ldots, t_n) \in \mathbb{C}^{n+1} \setminus \{O\}$.

Since $[\lambda t_0, \lambda t_1, \ldots, \lambda t_n] = [t_0, t_1, \ldots, t_n]$ for any $\lambda \in \mathbb{C}^\times$ and $(t_0, t_1, \ldots, t_n) \in \mathbb{C}^{n+1} \setminus \{O\}$, one sees that

$$[t_0, t_1, \ldots, t_n] = \left[1, \frac{t_1}{t_0}, \frac{t_2}{t_0}, \ldots, \frac{t_n}{t_0}\right], \quad \text{if } t_0 \neq 0.$$

-

-

-

Bibliography

Cox, D. A., Little, J. B. and O'Shea, D. (1997): *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Second Edition. xiv+536 pp., Undergraduate Texts in Mathematics. Springer-Verlag, New York, ISBN 0-387-94680-2 [An introductory textbook on computational commutative algebra and algebraic geometry. A standard reference work for students and those whose main interest may not be in mathematics proper.]

Cox, D. A., Little, J. B. and O'Shea, D. (1998): *Using Algebraic Geometry*. xii+499 pp., Graduate Texts in Mathematics 185. Springer-Verlag, New York, ISBN 0-387-98487-9; 0-387-98492-5 [An introductory textbook on computational algebraic geometry suitable for graduate students.]

Hartshorne, R. (1977): *Algebraic Geometry*. xvi+496 pp., Graduate Texts in Mathematics 52, Springer-Verlag, New York-Heidelberg, ISBN 0-387-90244-9[A standard graduate level textbook on algebraic geometry.]

Mumford, D. (1999): *The Red Book of Varieties and Schemes*. Second Expanded Edition. Includes the Michigan Lectures (1974) on Curves and Their Jacobians. With Contributions by Enrico Arbarello, x+306 pp., Lecture Notes in Mathematics 1358. Springer-Verlag, Berlin, ISBN 3-540-63293-X [One of the standard introductory textbooks on schemes. Combination of the author's Harvard and Michigan lecture notes.]

Mumford, D. (1995): *Algebraic Geometry, I. Complex Projective Varieties*. Reprint of the 1976 Edition, x+186 pp.,Classics in Mathematics, Springer-Verlag, Berlin, ISBN 3-540-58657-1[One of the standard textbooks on complex algebraic geometry.]

Reid, M. (1988): *Undergraduate Algebraic Geometry*. London Mathematical Society Student Texts 12. viii+129 pp., Cambridge University Press ISBN 0-521-35559-1; 0-521-35662-8. [This is an ideal self-contained introductory textbook on algebraic geometry at undergraduate level preparing the reader to the many good graduate level textbooks now available.]

Biographical Sketch

Tadao ODA, born 1940 in Kyoto, Japan

Education: BS in Mathematics, Kyoto University, Japan (March, 1962). MS in Mathematics, Kyoto University, Japan (March, 1964). Ph.D. in Mathematics, Harvard University, U.S.A. (June, 1967).

Positions held: Assistant, Department of Mathematics, Nagoya University, Japan (April, 1964-July, 1968) Instructor, Department of Mathematics, Princeton University, U.S.A. (September, 1967-June, 1968) Assistant Professor, Department of Mathematics, Nagoya University, Japan (July, 1968-September, 1975) Professor, Mathematical Institute, Tohoku University, Japan (October, 1975-March, 2003) Professor Emeritus, Tohoku University (April, 2003 to date)