AFFINE GEOMETRY, PROJECTIVE GEOMETRY, AND NON-EUCLIDEAN GEOMETRY

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Keywords: Affine geometry, projective geometry, non-Euclidean geometry, affine transformation, projective transformation, cross ratio, parallelism, collinearity, conic, perspective.

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Summary

Pictures on a TV screen are not real figures but projections onto a plane. A round ball in space is projected as a disc on the screen, but that disc will look like an oval when viewed from a slanted angle. On the other hand, a line segment is projected always as a line segment; even when viewed from a slanted angle, that projection never stops being a line segment. This difference between these two types of projections reflects a property of projections of three-dimensional objects into a two-dimensional plane. This chapter treats projective geometry, which explains properties of projections. It begins with the study of affine geometry, which is an intermediate between Euclidean geometry and projective geometry.

1. Affine Geometry

Affine geometry is a geometry studying objects whose shapes are preserved relative to affine transformations.

1.1. Affine Space

A real affine plane $\mathbb{A}^2$ is a plane equipped with the action of a two-dimensional vector space V over the real number field $\mathbb{R}$. It has an additive structure: for any point P in $\mathbb{A}^2$ and for any vector $\mathbf{v}$ in V, another point Q is determined by the rule $Q = P + \mathbf{v}$, and for any points P and Q there exists a unique vector $\mathbf{v}$ such that $Q = P + \mathbf{v}$. The vector $\mathbf{v}$ is denoted by $\overrightarrow{PQ}$. They satisfy compatibility with addition of vectors: $(P+\mathbf{u})+\mathbf{v} = P+(\mathbf{u}+\mathbf{v})$ for any two vectors $\mathbf{u}$ and $\mathbf{v}$. As a set, the plane $\mathbb{A}^2$ is identified with V itself. The addition $P \mapsto P+\mathbf{v}$ is called a *parallel transport* in the direction of $\mathbf{v}$. An n-dimensional affine space is defined likewise as a set equipped with an n-dimensional vector space. However, since the arguments for dimension two work generally for any dimension and since it is simpler to discuss two dimensional case, the description in the following is restricted to the latter.

The Euclidean plane, denoted usually by $\mathbb{R}^2$, has the structure of an affine plane together with a metric so that every vector $\mathbf{v}$ has a length.

Choose a point O in $\mathbb{A}^2$ and let $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2)$ be a basis of the vector space V. Then, any point P in $\mathbb{A}^2$ is written as $P = O + \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2$. The pair (λ_1, λ_2) is called the *affine coordinates* of the point P relative to the base point O and the basis $\mathbf{e}$. Choosing another point O' as a base point and another basis $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2)$, one can write $O' = O + a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{f}_i = \sum_{j=1}^{2} a_{ji}\mathbf{e}_j$. Then, the affine coordinates (λ'_1, λ'_2) of the point P relative to O' and $\mathbf{f}$ are given by the relation $\lambda_i = a_i + \sum_{j=1}^{2} a_{ji}\lambda'_j$.

1.2. Affine Lines

Let L be a one-dimensional subspace of V and P_0 a point in $\mathbb{A}^2$. Then the set $\{P_0 + \mathbf{v}; \mathbf{v} \in L\}$ is called an affine line through the point P_0 in the direction L, and can be denoted simply by ℓ. Through two points P and Q, there passes one and only one line, which is denoted by PQ and, hence, through any point, there passes a bundle of lines.

Two distinct lines are said to be *parallel* if and only if both lines have the same direction. They do not have common points. Let P, Q and R be three distinct points on a line. Then $\overrightarrow{PR} = r\overrightarrow{PQ}$ for some number r. This number is denoted by $\overrightarrow{PR}/\overrightarrow{PQ}$. If the affine plane is endowed with the Euclidean structure, this is equal to the ratio of the lengths of the line segments $\overline{PR}/\overline{PQ}$ provided Q and R are on the same side of P, and

$-\overline{PR}/\overline{PQ}$ provided Q and R are on opposite sides of P.

1.3. Affine transformations

An affine mapping is a pair (f,φ) such that f is a map from $\mathbb{A}^2$ into itself and φ is a *linear* map from V to V satisfying $f(P+\mathbf{v}) = f(P)+\varphi(\mathbf{v})$. It is called an affine transformation provided that the map f is bijective.

The image of a line under any affine transformation is a line.

Any number of points are said to be collinear when they lie on one line. The property above says that collinear points are mapped to collinear points by an affine transformation. An affine transformation is characterized as follows:

Let (f,φ) be an affine transformation. Then for any four points P, Q, R and S in the plane satisfying the relation $\overrightarrow{PQ} = r\overrightarrow{RS}$, the relation $\overrightarrow{f(P)f(Q)} = r\overrightarrow{f(R)f(S)}$ holds. Conversely, given a bijective map f with this property, one can define a linear map φ by $\varphi(\overrightarrow{PQ}) = \overrightarrow{f(P)f(Q)}$. Then (f,φ) defines an affine transformation.

In particular, the image of a line segment is again a line segment, and the midpoint of a line segment is mapped to the midpoint of the image of the line segment. Noting that the barycenter of a triangle is the intersection point of the three line segments from the vertices to the midpoints of the opposite edges, one also has the following statement:

The barycenter of a triangle is sent to the barycenter of the image of the triangle by an affine transformation.

Three points P, Q and R are said to be in general position if they are not collinear. The following property is called the homogeneity of the affine plane.

Let $\{P,Q,R\}$ and $\{A,B,C\}$ be two sets of three points in general position. Then there exists a unique affine transformation such that $f(P)=A$, $f(Q)=B$ and $f(R)=C$.

Let (f,φ) and (g,ψ) be two affine transformations. Then, the composition of mappings gives a new affine transformation $(f\circ g,\varphi\circ\psi)$. In fact, the set of all affine transformations, which we denote by G_{A}, has a group structure.

Choose a point O in $\mathbb{A}^2$ and a basis $\mathbf{e}=(\mathbf{e}_1,\mathbf{e}_2)$ of the vector space V, and let (f,φ) be an affine transformation. Write the vector $\overrightarrow{Of(O)}$ as $a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and the vector $\varphi(\mathbf{e}_i)$ as $a_{1i}\mathbf{e}_1 + a_{2i}\mathbf{e}_2$. Then one has an associated matrix

$$\Phi(f,\varphi)=\begin{pmatrix}1 & 0 & 0\\ a_1 & a_{11} & a_{12}\\ a_2 & a_{21} & a_{22}\end{pmatrix}. \qquad (1)$$

It is an exercise to show that $\Phi(f\circ g,\varphi\circ\psi)=\Phi(f,\varphi)\circ\Phi(g,\psi)$. This correspondence is called the matrix representation of the affine transformation. The part $\begin{pmatrix}a_1\\ a_2\end{pmatrix}$ parameterizes the space $\mathbb{R}^2$ and the sub-matrix $\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ belongs to the general linear group of degree 2.

1.4. Affine Collinearity

The following theorems are historically renowned ones on affine collinearity:

(Menelaus' theorem) *Let P, Q and R be three points in general position. Choose three points A, B and C on the lines QR, RP and PQ, respectively. Then A, B and C are collinear if and only if* $\overrightarrow{BP}/\overrightarrow{PC}\cdot\overrightarrow{CQ}/\overrightarrow{QA}\cdot\overrightarrow{AR}/\overrightarrow{RB}=-1$.

(Ceva's theorem) *Given a triangle ΔPQR, choose three points A, B and C on the lines QR, RP and PQ, respectively. Then, the three lines PA, QB and RC are all parallel or all meet at one point if and only if* $\overrightarrow{BP}/\overrightarrow{PC}\cdot\overrightarrow{CQ}/\overrightarrow{QA}\cdot\overrightarrow{AR}/\overrightarrow{RB}=1$.

Menelaus of Alexandria, who was born in about the year 70 in Alexandria and died in about 130, is famous for his work in spherical trigonometry. G. Ceva was born on 7 Dec 1674 in Milan and died on 15 June 1734 in Mantua, Italy.

1.5. Conic Sections

Given two lines ℓ and m intersecting at one point O in the three-dimensional Euclidean space, rotate the line m around ℓ. Then one gets a surface F called the *circular cone* with the vertex at O and the axis ℓ. The line m is called a generating line of F.

The set $F-\{O\}$ consists of two connected parts, F_1 and F_2. Let Π be a plane not passing through O. Then the section C of F by Π is called a *conic section*. If the conic section is bounded, then it is contained in either F_1 or F_2. Such a section is called an ellipse. If the section is not bounded but connected, then it is a parabola. If the section is neither bounded nor connected, then the section consists of two components and each component is called a hyperbola.

In Euclidean plane geometry, an ellipse is expressed as the set $\{(x,y)\in\mathbb{R}^2 \mid x^2/a^2+y^2/b^2=1\}$, where a and b are positive constants. A hyperbola is similarly expressed as $\{(x,y)\in\mathbb{R}^2 \mid x^2/a^2 - y^2/b^2 = 1\}$ and a parabola as $\{(x,y)\in\mathbb{R}^2 \mid y^2 = 4ax\}$, where $a>0$. In all three cases, the equations are quadratic. In this respect, the three kinds of curves are called quadrics (or conic curves).

From an affine-geometric point of view, any ellipse is regarded as the same: the ellipse expressed as $\{(x,y)\in\mathbb{R}^2 \mid x^2/a^2 + y^2/b^2 = 1\}$ is transformed to the circle $\{(x,y)\in\mathbb{R}^2 \mid x^2 + y^2 = 1\}$ by an affine transformation (f,φ), where f is the identity and $\varphi(x,y) = (x/a, y/b)$. Similarly, any hyperbola can be transformed to the rectangular hyperbola $\{(x,y)\in\mathbb{R}^2 \mid x^2 - y^2 = 1\}$ and any parabola to $\{(x,y)\in\mathbb{R}^2 \mid y^2 = 4x\}$. However, any ellipse, hyperbola, and parabola cannot be transformed to either of the other two types.

Apollonius of Perga, who was born in around 262 BC in Perga, Greece and died in about 190 BC in Alexandria, wrote a book called *Conics*, where terms such as parabola, ellipse, and hyperbola were introduced. It was J. Kepler (1571–1630) who showed that the orbits of planets around the sun form ellipses. Generally, due to kinematics, the motion of two planetary objects can be described by quadrics.

2. Projective Geometry

Projective geometry is a geometry studying objects whose shapes are preserved relative to projective transformations. A. Cayley (1821–1895) emphasized that “metrical geometry is a part of descriptive geometry (projective geometry in a modern terminology), and descriptive geometry is all geometry,” in relation to the discussion of the role of conics.

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Bibliography

D. Gans, *An introduction to non-Euclidean geometry*, Academic Press, N.Y. and London, 1973, ISBN:012274850-6. [Euclidean geometry, elliptic geometry and non-Euclidean geometry is discussed in an easily accessible fashion].

M. J. Greenberg, *Euclidean and Non-Euclidean Geometries, Development and History*, xv+400pp. W. H. Freeman and Co., San Francisco, 1973, ISBN:0-7167-1103-6. [This is one of the textbooks on the history of development from Euclidean geometry to non-Euclidean geometry].

J. V. Field, *Piero della Francesca. A mathematicians Art*, 256 pp. Yale University Press, New Haven, 2005, ISBN:0-300-10342-5. [This is a writing on a history of perspective drawing, beginning in the Italian Renaissance, and the life of Piero della Francesca].

H. Poincaré, *La Science et l'Hypothèse*, 298 pp. E. Flammarion, Paris, 1943. [A world classic, translated in several languages. See Chapter III Les géometries non euclidiennes].

Spaulding collection, Museum of Fine Arts at Boston: http://www.mfa.org. [This is one of the best collections of ukiyoe paintings].

O. Veblen and J. W. Young, *Projective geometry*, vol. 1 and 2, Blaisdell Publ. Co. Ginn and Co., N.Y.-Toronto-London, 1965, x+345 pp. and x+511 pp. [A standard monograph on projective geometry. The first versions were published in 1910 and 1917].

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