COMPLEX ANALYTIC GEOMETRY

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Summary

The main objective of Complex Analytic Geometry is to study the structure of complex manifolds and analytic varieties (the sets of common zeros of holomorphic functions). It is deeply related to various fundamental areas of mathematics, such as complex analysis, algebraic topology, commutative algebra, algebraic geometry, differential geometry and singularity theory, and there are very rich interplays among them. The Riemann-Roch theorem, for instance, is an outcome of such interaction. The subject is also related to many other branches of sciences including mathematical physics and learning theory.

1. Analytic Functions of One Complex Variable

Let U be an open set in the complex plane $\mathbb{C}$ and f a complex valued function on U. The differentiability of f at a point a in U is defined as in the case of functions of a real variable. Thus we say that f is *differentiable* at a if the limit

$$\lim_{h\to 0}\frac{f\left(a+h\right)-f(a)}{h}$$

exists. If f is differentiable at every point of U, we say that f is *holomorphic* in U. The above limit is denoted by $\frac{df}{dz}(a)$ and is called the derivative of f at a. If f is holomorphic in U, then we may think of $\frac{df}{dz}$ as a function on U.

We say that f is *analytic* at a point a in U if it can be expressed as a power series

$$f\left(z\right)=\sum_{n=0}^{\infty}c_n(z-a)^n,$$

which converges at each point z in a neighborhood of a. We say that f is analytic in U if it is analytic at every point of U.

If f is analytic in U, it is holomorphic in U. A striking fact about functions of a complex variable is that the converse is also true, *i.e.*, if f is holomorphic in U, it is analytic in U.

There is another important way of expressing this property. Let $z=x+\sqrt{-1}y$ with x and y the real and imaginary parts, respectively. We may think of f as a function of (x,y). We write $f=u+\sqrt{-1}v$ with u and v the real and imaginary parts. In general, we say that a function of real variables is (of class) C^r, if the partial derivatives exist up to order r and are continuous. If all the partial derivatives exist we say it is C^∞. Then f is holomorphic in U if and only if f is C^1 in (x,y) and satisfies the “Cauchy-Riemann equations" in U;

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

We finish this section by recalling the *Cauchy integral formula*. Let f be an analytic function in a neighborhood of a and γ the boundary of a small disk about a, oriented counterclockwise. Then we have

$$\frac{1}{2\pi\sqrt{-1}}\int_\gamma\frac{f(z)\,dz}{z-a}=f(a).$$

2. Analytic Functions of Several Complex Variables

Let $\mathbb{C}^n=\left\{z=(z_1,\ldots,z_n)\middle|z_i\in\mathbb{C}\right\}$ be the product of n copies of $\mathbb{C}$. For an n-tuple

$\nu = (\nu_1, \ldots, \nu_n)$ of non-negative integers, we set $z^\nu = z_1^{\nu_1} \ldots z_n^{\nu_n}$, $|\nu| = \nu_1 + \ldots + \nu_n$ and $\nu! = \nu_1! \ldots \nu_n!$.

Let U be an open set in $\mathbb{C}^n$ and f a complex valued function on U. We say that f is *analytic* at a point a in U, if it can be expressed as a power series

$$f(z) = \sum_{|\nu| \geq 0} c_\nu (z-a)^\nu = \sum_{\nu_1, \ldots, \nu_n \geq 0} c_{\nu_1 \ldots \nu_n} (z_1 - a_1)^{\nu_1} \ldots (z_n - a_n)^{\nu_n},$$

which converges absolutely at each point z in a neighborhood of a. We say that f is analytic in U if it is analytic at every point of U. A theorem of Hartogs says that f is analytic in U if and only if f is analytic in each variable z_i in U, for $i = 1, \ldots, n$.

In the sequel, we call analytic function also a holomorphic function and use the words “analytic" and “holomorphic" interchangeably. If f is holomorphic, for arbitrary ν, the partial derivative

$$\frac{\partial^\nu f}{\partial z^\nu} = \frac{\partial^{|\nu|} f}{\partial z_1^{\nu_1} \ldots \partial z_n^{\nu_n}}$$

exists and is holomorphic in U. If $f(z) = \sum c_\nu (z-a)^\nu$ is a power series expansion of f, then each coefficient c_ν is given by

$$c_\nu = \frac{1}{\nu!} \frac{\partial^\nu f}{\partial z^\nu}(a).$$

This series is called the Taylor series of f at a.

Let U be an open set in $\mathbb{C}^n$ and $f : U \to \mathbb{C}^m$ a map. We say that f is holomorphic if, when we write f componentwise as $f = (f_1, \ldots, f_m)$, each f_i is holomorphic. Let U and U' be two open sets in $\mathbb{C}^n$ and $f : U \to U'$ a map. We say that f is biholomorphic, if f is bijective and if both f and f^{-1} are holomorphic. For a holomorphic map $f = (f_1, \ldots, f_m)$ from an open set U in $\mathbb{C}^n$ into $\mathbb{C}^n$, we set

$$\frac{\partial(f_1, \ldots, f_m)}{\partial(z_1, \ldots, z_n)} = \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial z_1} & \cdots & \frac{\partial f_m}{\partial z_n} \end{pmatrix}$$

and call it the Jacobian matrix of f with respect to z.

We say that a point a in U is a *regular point* of f, if the rank of the Jacobian matrix evaluated at a is maximal possible, *i.e.*, $\min(n,m)$. Otherwise we say that a is a *critical* (or *singular*) point of f.

As in the case of functions or mappings of real variables, we have the inverse mapping theorem and the implicit function theorem, which are basic in analyzing a mapping at its regular point.

3. Germs of Holomorphic Functions

Let H be the set of functions holomorphic in some neighborhood of 0 in $\mathbb{C}^n$. We define a relation $\sim$ in H as follows. For two elements f and g in H, $f \sim g$ if they coincide on a neighborhood of 0. Then the relation $\sim$ is an equivalence relation in H. The equivalence class of a function f is called the *germ* of f at 0, which we also denote by f for simplicity. We let $\mathcal{O}_n$ be the quotient set of H by this equivalence relation. The set $\mathcal{O}_n$ has the structure of a commutative ring with respect to the operations induced from the addition and the multiplication of functions. It has the unity which is the equivalence class of the function constantly equal to 1.

If we denote by $\mathbb{C}\{z_1,\ldots,z_n\}$ the set of power series which converge absolutely in some neighborhood of 0, this set also has the structure of a ring. Since $f \sim g$ if and only if f and g have the same power series expansion, we may identify $\mathcal{O}_n$ with $\mathbb{C}\{z_1,\ldots,z_n\}$.

The ring $\mathcal{O}_n$ is an integral domain, *i.e.*, if $f\,g = 0$, for f, g in $\mathcal{O}_n$, then $f = 0$ or $g = 0$. We say that a germ u in $\mathcal{O}_n$ is a *unit* if there is a germ v such that $uv = 1$, it is equivalent to saying that it is the germ of a function u with $u(0) \neq 0$.

The following two theorems of Weierstrass are fundamental in the analysis of the structure of the ring $\mathcal{O}_n$. First, for a germ $f \neq 0$ in $\mathcal{O}_n$, we write $f = \sum_{|\nu|\geq 0} a_\nu z^\nu$. We say that the *order* of f is k, if $a_\nu = 0$ for all ν with $|\nu| < k$ and $a_{\nu_0} \neq 0$ for some ν_0 with $|\nu_0| = k$. We say that the order of f in z_n is k, if the order of $f(0,\ldots,0,z_n)$, as a power series in z_n, is k.

We consider the ring $\mathcal{O}_{n-1}[z_n]$ of polynomials in z_n with coefficients in $\mathcal{O}_{n-1}$:

$$\mathcal{O}_{n-1}[z_n] = \left\{ f(z) = a_0 + a_1 z_n + \ldots + a_k z_n^k \;\middle|\; a_i \in \mathcal{O}_{n-1} \right\}.$$

A *Weierstrass polynomial* in z_n of degree k is an element h of $\mathcal{O}_{n-1}\left[z_n\right]$ of the form

$$h = a_0 + a_1 z_n + \ldots + a_{k-1} z_n^{k-1} + z_n^k,$$

where k is a positive integer and $a_0, a_1, \ldots, a_{k-1}$ are non-units in $\mathcal{O}_{n-1}$.

Note that in the above, $h\left(0,\ldots,0,z_n\right) = z_n^k$. Hence the order of h in z_n is k. In general, any germ f in $\mathcal{O}_n$ is written as

$$f(z) = a_0 + a_1 z_n + \ldots + a_k z_n^k + \cdots$$

with a_i in $\mathcal{O}_{n-1}$. The order of f in z_n is k if and only if $a_0, a_1, \ldots, a_{k-1}$ are non-units in $\mathcal{O}_{n-1}$ and a_k is a unit in $\mathcal{O}_{n-1}$. In this case, $a_k^{-1}\left(a_0 + a_1 z_n + \ldots + a_k z_n^k\right)$ is a Weierstrass polynomial in z_n of degree k. The Weierstrass preparation theorem stated below says that such an f is essentially equal to a Weierstrass polynomial of degree k.

Weierstrass division theorem

If h is a Weierstrass polynomial in z_n of degree k, then for any germ f in $\mathcal{O}_n$, there exist uniquely determined elements q in $\mathcal{O}_n$ and r in $\mathcal{O}_{n-1}\left[z_n\right]$ with $\deg r < k$ such that

$$f = qh + r.$$

Weierstrass preparation theorem

Let f be a germ in $\mathcal{O}_n$ whose order in z_n is k. Then there is a unique Weierstrass polynomial h in z_n of degree k such that $f = uh$ with u a unit in $\mathcal{O}_n$.

Next we discuss some important properties of the ring $\mathcal{O}_n$ which follow from the above theorems. We say that a germ f in $\mathcal{O}_n$ is *irreducible* if f is not a unit and if the identity $f = gh$ for germs g and h implies that either g or h is a unit. The ring $\mathcal{O}_n$ is a unique factorization domain, *i.e.*, every germ f that is not 0 or a unit can be expressed as a product of irreducible germs and the expression is unique up to the order and multiplications by units. For germs f and g, there is always the greatest common divisor $\gcd\left(f,g\right)$, which is unique up to multiplication by units. We say that f and g are *relatively prime* if $\gcd\left(f,g\right)$ is a unit.

Another important property of the ring $\mathcal{O}_n$ is that it is *Noetherian*, *i.e.*, every ideal in $\mathcal{O}_n$ has a finite number of generators.

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Biographical Sketch

Tatsuo SUWA: The main subject of the author's research is Complex Analytic Geometry. He started his career under the guidance of K. Kodaira. Earlier works are on the structures and deformations of compact complex manifolds, such as some complex surfaces, compact quotients of the complex Euclidean spaces and holomorphic Seifert fiber spaces.

Then, inspired by a paper of P. Baum and R. Bott, published in 1972, he became interested in singular holomorphic foliations (integrable system of holomorphic vector fields or 1-forms with singularities). He first constructed an unfolding theory of codimension one foliation. This generalizes the unfolding theory for functions, which is developed by J. Mather and is deeply related to the catastrophe theory of R. Thom. One of the basic results of the author's is a "versality theorem", which gives an algebraic criterion for an unfolding to be versal and has many applications.

He then started work on the residues of singular holomorphic foliations. With a number of collaborators, J.-P. Brasselet, D. Lehmann and J. Seade to name a few, he generalized the index theorem of C. Camacho and P. Sad for invariant curves of foliations and discovered some new indices and residues. He constructed a residue theory, which unify the Baum-Bott, Camacho-Sad and the other theories, as a localization theory of some characteristic classes in the framework of Cech-de Rham cohomology. This was published as a book from Hermann, Paris in 1998.

This localization theory turned out to be very effective in dealing with many problems involving characteristic classes, including the characteristic classes of singular varieties (for which he constructed a theory of Milnor classes with the above collaborators), residues of Chern classes and explicit representations of these, analytic intersection theory in singular varieties, and applications to complex dynamical systems (for which he made contributions with F. Bracci et al.).

The author has been invited to many countries for collaborations and for lectures, including Brazil, Canada, China, France, Germany, Hungary, Italy, Korea, Mexico, Poland, Russia, Spain, Tunisia, U.S.A. He also invited researchers from all over the world and organized a number of international symposia. In particular, he organized three times, with his partner J.-P. Brasselet, Franco-Japanese symposia with the title "Singularities in Geometry and Topology".