FUNCTIONAL ANALYSIS AND FUNCTION SPACES

Mikihiro HAYASHI
Department of Mathematics, Hokkaido University, JAPAN

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Contents

Summary

Fundamental concepts and terminologies in functional analysis are explained. In addition, some typical function spaces are listed with their basic properties. Knowledge of linear algebras will help in the understanding of this chapter. A nontrivial function space is an infinite dimensional vector space with a geometric structure in various types. Functional analysis offers tools and terminologies to treat such function spaces systematically. Knowledge of the general topology is partially assumed to read some subsections. In terms of nets, a generalized concept of sequences, it may be possible to feel at least a flavor of the present subjects by making analogy to the known case. A basic knowledge of the Lebesgue integration theory and the complex function theory may be helpful for some parts.

1. Introduction

In analysis there are two standard methods to investigate functions. (A function always means a real–valued or a complex–valued function in this chapter.) The first method is classical. One investigates properties of functions individually by calculating values, derivatives, integrals, or by drawing graphs. The second method is modern. One considers a set of functions, regards each function as a point in the set, and investigates geometric and algebraic structures of the set. The latter method is called **functional analysis**, while the former method is called **classical analysis.**

A set of functions under consideration is called a **function space**. A mapping from a function space to a function space is called an **operato**r. A function on a function space is called a **functional**. For example, the mapping which relates a function x to its derivative dx/dt is an operator and the function which relates a function x to a definite integral $\int_a^b x(t)\,dt$ is a functional. It is a major subject in functional analysis to investigate properties of operators, functionals and function spaces.

A primitive idea on the use of functional analysis is easy to understand by the following example.

Volterra's integral equation: Find an unknown function $x(s)$ satisfying

$$y(s) = x(s) - \int_a^s K(s,t)\,x(t)\,dt \tag{1}$$

for a given continuous function $y(s)$, where $K(s,t)$ is a continuous function of two variables s and t for $a \le t \le s \le b$. Let X be the set of all continuous functions on an interval $[a,b]$ and define two operators T and I from X to itself by $(Tx)(s) = \int_a^s K(s,t)\,x(t)\,dt$ and $(Ix)(s) = x(s)$, respectively. Here the notation Tx and Ix represents functions of s given by the right hand side of the equations on the one hand, and represents the idea that the function x is mapped by T and I to the functions Tx and Ix respectively, on the other. Eq.(1) is then written as $y = Ix - Tx = (I-T)x.$ If $I-T$ admits an inverse mapping $(I-T)^{-1}$, the equation has a unique solution $x = (I-T)^{-1}y$. Consider the identity:

$$(I-T)\left(I + T + T^2 + \ldots + T^{n-1}\right)y = \left(I - T^n\right)y \tag{2}$$

where $T^n y = T\left(T^{n-1}y\right)$. If the series

$$x = y + Ty + \cdots + T^n y + \cdots \tag{3}$$

converges, then $T^n y \to 0 (n \to \infty)$ and Eq.(2) yields

$$(I-T)^{-1} = I + T + T^2 + \cdots + T^n + \cdots. \tag{4}$$

The solution of (1) is given by the series (3).

The above argument contains an ambiguity; it is not properly mentioned in what fashion the series (3) and (4) converge. (Meanwhile, an argument with ambiguities like above is often useful as a heuristic method). There are many different ways of convergence considered for sequences of functions and several types for sequences of operators. As for the series (3), its uniform convergence follows from the inequality.

$$\left|(T^n x)(s)\right| \le \frac{M^n (s-a)^n}{n!} L \quad (a \le s \le b), \tag{5}$$

where M and L are constants with $|K(s,t)| \le M$ and $|g(s)| \le L$ for $a \le t \le s \le b$. The inequality (5) itself is proved by means of classical analysis. (It is worthwhile to note that the classical analysis is lying in the background of functional analysis and often playing a crucial role at a key point.)

Analysis becomes more powerful if limits of functions are realized as elements in a function space under consideration. Even if one starts from a sequence of polynomials, different ways of convergence lead to different function spaces. In some cases, the notion of functions has to be generalized in order to realize limit elements of functions. Some function spaces were introduced to define an operator on it; most operators appearing in applications require their own function spaces, on which the operator behaves properly. Some function spaces, defined on a set *S* of their variables, are utilized to investigate the structure of the base space *S*; the geometric structure of the base space *S* reflects itself to certain types of function spaces. Some function spaces are studied to investigate functions themselves; certain properties of functions appear in relations with other functions. This is why we have a variety of function spaces. It is impossible to list all useful function spaces in this chapter; only a very limited number of typical function spaces are listed in Section 5

In order to study various function spaces and operators on them systematically, abstract treatments have been developed and are systematized as the theory of functional analysis. Function spaces form an infinite dimensional vector space, in which a way of convergence is defined. Nontriviality in functional analysis shows itself when one considers convergence in an infinite dimensional vector space. When a vector space is finite dimensional, the subjects reduce to the theory of linear algebras; a (linear) operator on it is nothing but a matrix. Basic concepts in functional analysis are treated in Sections 3 and 4.

Around the beginning of the twentieth century the theory of functional analysis began to be developed by V.Volterra, I Fredholm, and some others in connection with the study of integral equations. Fundamental concepts in functional analysis have been founded by D. Hilbert, F. Riesz, S. Banach, J. von Neumann, the Bourbaki group, I.M. Gelfand, and others in the first half of the twentieth century. Functional analysis is now developing in connection with many areas of mathematics, such as functional equations, the operator theory, Fourier analysis, the complex function theory, etc.

2. Function Spaces and Some Examples

Let S be a nonempty set. Denote by $\mathcal{F}(S)$ the set of all scalar–valued functions on S, where the scalars are either the real numbers or complex numbers.

For functions x, y on S and a scalar c, new functions cx and $x+y$ on S are defined by the **pointwise operations** $(cx)(s) := cx(s)$ and $(x+y)(s) := x(s)+y(s)\,(s \in S)$. The set $\mathcal{F}(S)$ forms a vector space with these two pointwise operations. In this chapter, a vector subspace of $\mathcal{F}(S)$ will be called a **function space** on S. For instance, the set $\mathcal{FB}(S)$ of all bounded functions on S is a function space; a function x is said to be **bounded** if $\|x\|_S = \sup_{s \in S} |x(s)|$ is finite. In other words, for each function x in this function space, there is an upper bound K for its absolute value, satisfying $|x(s)| \leq K$ for all $s \in S$ where K depends on x. The quantity $\|x\|_S$ is called the **sup-norm** (or **supremum norm** or **uniform norm**) of the function x on the set S. A subset X of a vector space is a vector subspace if (and only if) X is closed under two operations, addition and scalar multiplication, i.e., cx and $x+y$ belong to X for any $x, y \in X$ and any scalar c. (It is much easier to show a subset X of $\mathcal{F}(\mathcal{S})$ to be a vector subspace than to show directly that X is a vector space by checking all the axioms of a vector space.)

Let $S = \{1, 2, \ldots, N\}$. Identifying a function x on S with a point $(x(1), x(2), \ldots, x(N))$ in $\mathbf{R}^N$ $(\mathbf{C}^N)$, the N-dimensional real (complex) space $\mathbf{R}^N$ $(\mathbf{C}^N)$ may be regarded as the function space $\mathcal{F}(\{1, 2, \ldots, N\})$. Similarly, a function x on the set $\mathbf{N}= \{1, 2, 3,\ldots\}$ can be identified with the sequence $(x(1), x(2), x(3), \ldots)$. Thus, the set of all sequences of scalars may be regarded as the function space $\mathcal{F}(\mathbf{N})$.

Here are some examples of function spaces on an interval J in the real line $\mathbf{R}$ with positive or infinite length.

$\mathcal{C}(J)$: the set of all continuous functions on J.

$\mathcal{CB}(J)$ the set of all bounded continuous functions on J.

$\mathcal{L}^p(J)(0<p<\infty)$: the set of all measurable functions x on J such that $\int_J |x(s)|^p ds < \infty$.

$\mathcal{L}^\infty(J)$: the set of all measurable functions x on J such that $x(s)$ is essentially bounded (i.e, there is a constant M such that $|x(s)| \leq M$ holds almost everywhere for $s \in J$).

With the convention that two functions are equivalent (and identified with each other) if they are equal almost everywhere, $L^p(J)$ denotes the set of equivalence classes of functions in $\mathcal{L}^p(J)(0<\mathrm{p}\leq\infty)$. (In some literature, the symbol $\mathcal{L}^p$ is used to represent both the function space $\mathcal{L}^p$ and the set L^p of equivalence classes.)

3. Basic Concepts in Functional Analysis

3.1. Normed spaces and Banach Spaces

Let X be a real or complex vector space. A real–valued function $x \to \|x\|$ on X is called a **norm** on X, if it satisfies the following four properties for any elements x, y in X and any scalar c: i) $\|x\| \geq 0$; ii) $\|x\| = 0$ if only if $x = 0$; iii) $\|cx\| = |c|\,\|x\|$; iv) $\|x+y\| \leq \|x\| + \|y\|$. A vector space X equipped with a norm is called a **normed vector space**. The quantity $\|x\|$ is called the **norm** of the element x. The norm $\|x-y\|$ of $x-y$ is called the **distance** between two elements x and y. Replacing x, y in the inequality (iv) by $x-z,\ z-y$, respectively, it follows that $\|x-y\| \leq \|x-z\| + \|z-y\|$, which is the **triangle inequality.**

A sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$ is denoted by $\{x_n\}_{n=1}^\infty$ or $\{x_n\}$. A sequence $\{x_n\}$ in a normed vector space X is said to **converge** (or more precisely, **converge strongly**) to an element x if the sequence $\{\|x_n - x\|\}$ of real numbers converges to zero; this is written notationally as $\lim x_n = x$ (or $s-\lim x_n = x$) and the element x is said to be the **limit** (or **strong limit**) of the sequence $\{x_n\}$. The modifier 'strongly' or 'strong' may be omitted unless one needs to distinguish the convergence in the above sense from others. (In functional analysis, several types of convergence are introduced in various ways. See Section 4.2)

A subset of E or X is said to be **closed** if every limit of converging sequence in E belongs to E. A subset of a normed vector space X is said to be **bounded** if it is contained in a ball $\|x\| \leq R$ of a finite radius R. A subset K of a normed vector space X is **compact** if and only if every sequence $\{x_n\}$ in K has a converging subsequence and its limit point is in K. Every compact subset of a normed vector space is closed and bounded. If a closed ball $\{x \in X : \|x\| \leq R\}(R > 0)$ is compact in a normed vector space X, then X is the finite dimensional. In particular, the **Heine-Borel theorem** stating that every bounded closed subset is compact is no longer true for an infinite dimensional normed vector space X.

If a sequence $\|x_m - x_n\|$ tends to zero as both m and n go to infinity, then the sequence $\{x_n\}$ is called a **Cauchy sequence**. A converging sequence is a Cauchy sequence. A normed vector space is said to be c**omplete** if every Cauchy sequence converges. A complete normed vector space is called a **Banach Space**.

Let X and Y be normed vector spaces. The direct product $X \times Y := \{(x, y) : x \in X, y \in Y\}$ of X and Y forms a normed vector space with the addition $(x, y) + (x', y') := (x + x', y + y')$ the scalar multiplication $c(x, y) := (cx, cy)$ and the norm $\|(x, y)\| := \|x\| + \|y\|$. If X and Y are Banach spaces, then $X \times Y$ is also a Banach space. If N is a closed vector subspace N of X, then the quotient space $X / N := \{x + N : x \in X\}$ is a normed vector space with the addition $(x + N) + (y + N) := (x + y) + N$, the scalar multiplication $c(x + N) := (cx) + N$ and the norm $\|x + N\| := \inf\{x + z : z \in N\}$, where $x + N := \{x + z : z \in N\}$, called the coset containing the element x. If X is a Banach space, then X / N is also a Banach space.

The following vector spaces with the indicated norms are Banach spaces.

$\mathbf{R}^N, \mathbf{C}^N$: $\|x\| = \left(|x_1|^2 + \cdots + |x_N|^2\right)^{1/2}, x = (x_1, \ldots, x_N)$.

$\left(\ell^p\right) = \left\{(x_1, x_2, x_3, ...) : \sum_{j=1}^{\infty} |x_j|^p < \infty\right\}$ $(1 \leq p < \infty)$: $\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}, x = (x_1, x_2, x_3, ...)$.

$\left(\ell^\infty\right) = \{(x_1, x_2, x_3, ...)$: $\{x_j\}$ is a bounded sequence $\}$: $\|x\|_\infty = \sup_j |x_j|, x = (x_1, x_2, x_{3,} \ldots)$.

$(c_0) = \left\{(x_1, x_2, x_3, ...) : \lim_{j \to \infty} x_j = 0\right\}$: $\|x\|_\infty = \sup_j |x_j|, x = (x_1, x_2, x_3, ...)$.

$(c) = \{(x_1, x_2, x_3, ...)$: $\{x_j\}$ is a converging sequence}: $\|x_\infty\| = \sup_j |x_j|, x = (x_1, x_2, x_3, ...)$.

$\mathcal{CB}(J): \|x\|_J = \sup_{s\in J} |x(s)|$, the sup-norm on a nonempty interval J.

$L^p(J)\ (1 \le p < \infty)$: $\|x\|_p = \left(\int_J |x(s)|^p\, ds\right)^{1/p}$.

$L^\infty(J): \|x\|_\infty = \text{ess-}\sup_{s\in J} |x(s)| = \inf\{M : |x(s)| \le M$ (almost everywhere)}. This norm is called the **essential supermum norm**

A normed vector space X is said to be **separable** if X has a dense countable subset (i.e., a countable subset $\{x_1, x_2, ...\}$ such that every element of X is the limit of some subsequence of $\{x_n\}$). The Banach spaces $(\ell^p)\ (1 \le p < \infty)$, (c_0), (c), and $L^p(J)\ (1 \le p < \infty)$ are separable, and the Banach spaces (ℓ^∞) and $L^\infty(J)$ are not separable. The Banach space $\mathcal{CB}(J)$ is separable if J is a bounded closed interval $[a,b] = \{t : a \le t \le b\}$, and $\mathcal{CB}(J)$ is not separable if J is any other type of nonempty interval.

A **completion** of $\tilde{X}$ of a normed vector space X is a Banach space such that X is identified with a dense subspace of $\tilde{X}$. Such a completion $\tilde{X}$ always exists and is uniquely determined by X up to isometric isomorphisms of $\tilde{X}$ (see Section 3.3 for 'isometric isomorphism'). In many cases, the completion of a function space turns out to be a function space. In a general theory of completion, however, the elements of completion $\tilde{X}$ are idealistic; they are defined as equivalence classes of Cauchy sequences. Therefore, one may often need another theory of its own in order to describe the elements of completion on a substantial way. For instance, the completion of the function space $\mathcal{C}([0,1])$ with the norm $\|x\|_p$ is the Banach space $L^p([0,1])$; the Lebesgue integration theory is used to describe the elements of $L^p([0,1])$.

3.2. Hilbert Spaces

An **inner product** on a vector space H is a scalar–valued function (x,y) of x and y in H with the following five properties: i) $(x,x) \ge 0$; ii) $(x,x) = 0$ if only if $x = 0$; iii) $(cx,y) = c(x,y)$ (c: any scalar); iv) $(x+y,z) = (x,z) + (y,z)$ $(x,y,z \in H)$); v) $(\overline{x,y}) = (y,x)$ ($(x,y) = (y,x)$ if the scalars are real). A vector space H equipped with an inner product is called an **inner product space** (or **metric vector space** or **pre-Hilbert space**). An inner product space H is a normed vector space with a norm defined by $\|x\| := \sqrt{(x,x)}$ and the **Cauchy-Schwarz** inequality $|(x,y)| \le \|x\|\,\|y\|$ is valid. An inner product space H is called a **Hilbert space** if H is complete as a normed vector space. The completion of an inner product space as a normed space is a Hilbert space; more precisely, the inner product on a pre-Hilbert space can be uniquely extended to its completion. The norm $\|x\| := \sqrt{(x,x)}$ satisfies the equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|. \qquad (6)$$

Conversely, a normed vector space X can be made an inner product space with $\|x\| = \sqrt{(x,x)}$ if only if its norm $\|x\|$ satisfies the equality (6); in this case, an inner product is defined by $(x,y) := \left(\|x+y\|^2 - \|x-y\|^2\right)/4$ if the scalars are real and by $(x,y) := \left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)/4$ if the scalars are complex.

Let H be a Hilbert space. Two elements x and y in H are said to be mutually **orthogonal** if $(x,y)=0$. A family $\{e_i\}$ of elements in H is said to be **orthonormal set** (or **system**) if $\|e_i\|=1$ and $(e_i,e_j)=0$ for $i \neq j$. Any orthonormal set $\{e_i\}$ satisfies Bessel's inequality $\|x\|^2 \geq \sum_i |(x,e_i)|^2$. A maximal orthonormal set $\{e_i\}$ is said to be **complete**. A complete orthonormal set is also called an **orthonormal basis**. For an orthonormal set $\{e_i\}$, the following three properties are mutually equivalent; i) $\{e_i\}$ is complete; ii) **Parseval's equality** $\|x\|^2 = \sum_i |(x,e_i)|^2$ holds for every x in H,; iii) **Fourier expansion** $x = \sum_i (x,e_i)e_i$ holds for every x in H. (The number of the elements in e_i can be uncoutably infinite. Even in that case, at most a countable number of nonzero terms remain in the sums of Parsevel's equality and Fourier expansion, and the sums converge.) if $\{f_i\}$ is another orthonormal basis, then there is a one-to–one correspondence between $\{e_i\}$ and $\{f_i\}$, i.e., any orthonormal basis of H consists of the same (cardinal) number of elements, and the (cardinal) number is called the **dimension** of Hilbert space H. (Note that the dimension of an infinite dimensional Hilbert space is different from the algebraic dimension as a vector space; the latter is the (cardinal) number of a basis of H as a vector space. Two types of dimensions coincide with each other if one is finite)

The following vector spaces are Hilbert spaces with the indicated inner products.

$\mathbf{R}^N$ or $\mathbf{C}^N$: $(x,y) = x_1\overline{y}_1 + \cdots + x_N\overline{y}_N$.

(ℓ^2): $(x,y) = \sum_{j=1}^{\infty} x_j\overline{y}_j$.

$L^2(J)$: $(x,y) = \int_J x(s)\overline{y(s)}\,ds.$

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Bigraphical Sketch

Mikihiro Hayashi was born on September 1, 1948. He received BS, MS and PhD degrees in 1971, 1973, and 1979 from Saitama University, Hokkaido University and University of California at Los Angeles, respectively. During 1974 April -- 1982 March, he was an Instructor at Ibaraki University. From 1982 April to 1987 March, he was a Lecturer at Hokkaido University and became an Associate Professor in 1987 and Professor in May 1993.