MODEL THEORY

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Contents

Summary

Model theory uses formal languages to study mathematical structures. Until about 1970, the main focus of research was in *classical model theory*, where the language of first order logic is used to study the class of all mathematical structures.

Since then, the subject has split up into many branches, which have different objectives, separate conferences, and almost disjoint sets of researchers. We will try to describe the current landscape from a broad perspective, with brief overviews of several branches of model theory.

1. Introduction

Among the earliest results in model theory were the theorems of Löwenheim and Skolem, Tarski's work on truth and definability, and Gödel's completeness theorem. The subject became recognizable as an area of research around 1950, with the applications of the compactness theorem to algebra by Henkin, Tarski, Malcev, and Abraham Robinson.

Section 2 contains an outline of classical model theory— the model theory of first order logic circa 1970. More recently, the subject has moved along three very general paths: (A) Use first order logic to study models of "tame" theories—theories whose models can be described by invariants of some kind. (B) Start with a formal language and study the structures which interpret the language. (C) Start with the class of structures arising in a specific area of mathematics and build a formal language which is appropriate for the study of those structures. In Sections 3, 4, and 5 we give summaries of some work along each of these three paths.

Model theory along path (A) includes stable and simple theories, o-minimal theories, and computable model theory. Applications of model theory to algebra are usually found along this path.

Along path (B) one finds the model theory of infinitary logics and admissible sets, logic with extra quantifiers, and modal model theory.

Along path (C) are finite model theory, model theories for topological structures, Banach spaces, and stochastic processes. Applications of model theory to computer science and to analysis are often found on this path.

Throughout this chapter, $\kappa, \lambda, \ldots$ denote infinite cardinals, and the cardinality of a set X is denoted by $|X|$. Cardinals are defined as initial ordinals, so that $\omega = \aleph_0$, and ω_α is both the α-th cardinal $\aleph_\alpha$ and the smallest ordinal of size $\aleph_\alpha$. For finite tuples we write $\vec{x} = (x_1, \ldots, x_n)$, and $|\vec{x}| = n$. We will assume throughout that L is a countable vocabulary consisting of relation, function, and constant symbols, always including the equality symbol $=$. We keep L countable to simplify the exposition, but some of the results we state have analogues for uncountable vocabularies.

2. Classical Model Theory

We refer to the general introduction on *Formal Logic* for the basics of first order logic, including the notions of formula, sentence, model, truth value, theory, and complete theory, the various uses of the satisfaction symbol $\vDash$, and the completeness theorem. Throughout this section, formulas and theories are understood to be first-order with a countable vocabulary L. The interpretations of first order logic, which we will call **first order structures** to distinguish them from other kinds of structures, consist of a universe set M and a relation, function, or constant on M corresponding to each symbol of L. $\mathcal{M}, \mathcal{N}, \ldots$ will always denote first order structures for L with universe sets $M, N, \ldots$.

We review the basic theorem which forms the starting point for classical model theory.

Theorem 2.0.1. (*Compactness and Löwenheim-Skolem Theorem*) *Let* Γ *be a set of sentences*. *If every finite subset of* Γ *has a model, then* Γ *has a model of cardinality at most* ω.

(This result still holds for uncountable vocabularies, but in that case Γ has a model of cardinality at most $\max(\omega, |\Gamma|)$.

Lindström proved a striking converse result. For a rigorous statement of the result, one must define the general notion of a logic $\mathbb{L}$ and of a model of a sentence of $\mathbb{L}$. We omit the formal definition, which is a long list of obvious properties.

Theorem 2.0.2. (*Lindström's Theorem*) *First order logic is the maximal logic* $\mathbb{L}$ *such that:*
(*a*) *The interpretations of* $\mathbb{L}$ *are first order structures.*
(*b*) (*Countable compactness property*) *If* Γ *is a countable set of sentences of* $\mathbb{L}$ *and every finite subset of* Γ *has a model, then* Γ *has a model.*
(*c*) (*Löwenheim property*) *Each sentence of* $\mathbb{L}$ *which has a model has a model of cardinality at most* ω.

There are several other characterizations of first order logic in a similar vein; for a survey of these matters see the collection.

2.1. Constructing Models

In this subsection we describe some basic methods of constructing models of first order theories. The first of these methods, called the method of diagrams, was introduced by Henkin and Robinson, in order to prove the completeness theorem and other results. The diagram of a structure is a generalization of the multiplication table of a group.

An **expansion** of $\mathcal{M}$ is a structure $\mathcal{M}'$ for a larger vocabulary $L' \supseteq L$ such that $\mathcal{M}'$ has the same universe as $\mathcal{M}$ and each symbol of L has the same interpretation in $\mathcal{M}'$ as in $\mathcal{M}$.

Given $X \subseteq M$, L_X is the vocabulary formed by adding a new constant symbol c_x to L for each $x \in X$, and $\mathcal{M}_X = (\mathcal{M}, x)_{x \in X}$ is the expansion of $\mathcal{M}$ formed by interpreting each c_x by x. The **diagram** of $\mathcal{M}$ is the set of all atomic and negated atomic sentences true in $\mathcal{M}_M$, and the **elementary diagram** of $\mathcal{M}$ is the set $Th(\mathcal{M}_M)$ of all sentences true in $\mathcal{M}_M$.

A function $f : M \to N$ is an isomorphic embedding of $\mathcal{M}$ into $\mathcal{N}$ if and only if $(\mathcal{N}, fa)_{a \in M}$ is a model of the diagram of $\mathcal{M}$.

The important notion of an elementary extension was introduced by Tarski and Vaught.

Two structures $\mathcal{M}$ and $\mathcal{N}$ are **elementarily equivalent**, in symbols $\mathcal{M} \equiv \mathcal{N}$ or $Th(\mathcal{M}) = Th(\mathcal{N})$, if they satisfy the same sentences of first order logic. $\mathcal{N}$ is an **elementary extension of** $\mathcal{M}$, in symbols $\mathcal{M} \prec \mathcal{N}$, if $M \subseteq N$ and $\mathcal{M}_M \equiv \mathcal{N}_M$. A function $f : M \to N$ is an **elementary embedding** of $\mathcal{M}$ into $\mathcal{N}$, in symbols $f : \mathcal{M} \prec \mathcal{N}$, if $(\mathcal{N}, fa)_{a \in M}$ is a model of the elementary diagram of $\mathcal{M}$. Note that $f : \mathcal{M} \prec \mathcal{N}$ implies $\mathcal{M} \equiv \mathcal{N}$.

The first application of the method of diagrams was the completeness theorem. Another variant of the method gives the following.

Theorem 2.1.1. (*Downward Löwenheim–Skolem–Tarski*) *For every infinite* $X \subseteq \mathcal{N}$, *there exists* $\mathcal{M} \prec \mathcal{N}$ *such that* $X \subseteq M$ *and* $|M| = |X|$.

Theorem 2.1.2. (*Upward Löwenheim-Skolem-Tarski*) *If* $\omega \leq |M| \leq \kappa$, *then* $\mathcal{M}$ *has a proper elementary extension of cardinality* κ.

When we write a set of formulas in the form $\Gamma(\vec{v})$, it is understood that $\vec{v}$ is a finite tuple which contains every free variable occurring in Γ. The notation $\mathcal{M} \models \Gamma[\vec{a}]$ means that the tuple $\vec{a}$ satisfies every formula in the set $\Gamma(\vec{v})$. We say that $\mathcal{M}$ **realizes** $\Gamma(\vec{v})$ if $\mathcal{M} \models \Gamma[\vec{a}]$ for some tuple $\vec{a}$ in M, and that $\mathcal{M}$ **omits** $\Gamma(\vec{v})$ otherwise. The **type** $tp(\vec{a}/X)$ of a tuple $\vec{a}$ over a set $X \subseteq M$ (in $\mathcal{M}$) is the set of all formulas $\varphi(\vec{v})$ of L_X satisfied by $\vec{a}$ in $\mathcal{M}_X$.

The Omitting Types Theorem is an important application of the method of diagrams. It says that if there is no single formula which is consistent with T and together with T implies Γ, then T has a countable model which omits Γ. In fact, the result holds for countably many sets of formulas.

Theorem 2.1.3. (*Omitting Types*) *Let* T *be a consistent theory* (*not necessarily complete*), *and for each* $k < \omega$ *let* $\Gamma_k(\vec{v}_k)$ *be a set of formulas*. *Then either*

(*i*) T *has a countable model* $\mathcal{M}$ *which omits each of the sets* $\Gamma_k(\vec{v}_k)$, *or*

(*ii*) *For some* $k < \omega$, *there is a formula* $\varphi(\vec{v}_k)$ *which is consistent with* T *such that* $T \models (\forall \vec{v}_k)[\varphi(\vec{v}_k) \to \psi(\vec{v}_k)]$ for all $\psi \in \Gamma_k$.

A theory T is said to be κ**-categorical** if it has infinite models and any two models of T of cardinality κ are isomorphic. The Löwenheim-Skolem-Tarski theorems have the following corollary.

Corollary 2.1.4. (*Łos-Vaught test*) *Any theory which has no finite models and is* κ*-categorical for some* κ *is complete*.

Sometimes a theory is described in the following way. Given a class K of structures, the **theory of** K is defined as the set of all sentences true in all $\mathcal{M} \in K$. By a **set of axioms** for a theory T we mean a set of sentences which is logically equivalent to T.

Examples: The theory of infinite Abelian groups with all elements of order p is κ-categorical for all κ.

The theory of algebraically closed fields of given characteristic is ω_1-categorical but not ω-categorical.

The theory of dense linear order, and the theory of ∞ by ∞ equivalence relations (infinitely many classes, all infinite), are ω-categorical but not ω_1-categorical.

By the Łos-Vaught test, each of these theories is complete.

The following two results are consequences of the Omitting Types Theorem.

Theorem 2.1.5. (*Ryll-Nardzewski*, *Engeler*, *Svenonius*) *A complete theory T with infinite models is ω-categorical if and only if for each $n < \omega$ there are only finitely many formulas in n free variables which are nonequivalent with respect to T.*

Theorem 2.1.6. (*VaughtV1*) *There is no complete theory which has exactly two countable models* (*up to isomorphism*).

For each $3 \leq n < \omega$, Ehrenfeucht gave an example of a complete theory with exactly n countable models. The complete theory of (ω, S) where S is the successor function, is an example with ω countable models. "Most" complete theories (for example, the complete theory of $(\omega, +, *)$) have the maximum number 2^ω of countable models.

The following conjecture is still open and has stimulated a great deal of research.

Conjecture 1. (*Vaught*) *There is no complete theory with more than ω and fewer than 2^ω countable models*.

Of course, this conjecture is trivial under the continuum hypotheses. Here is a partial result.

Theorem 2.1.7. (*Morley*) *There is no complete theory with more than ω_1 and fewer than 2^ω countable models*.

An **elementary chain** is a sequence of structures $(\mathcal{M}_n, n < \omega)$ such that $\mathcal{M}_n \prec \mathcal{M}_{n+1}$ for each n.

Theorem 2.1.8. (*Tarski and Vaught*) *If $(\mathcal{M}_n, n < \omega)$ is an elementary chain, then for each k, $\mathcal{M}_k \prec \bigcup_n \mathcal{M}_n$.*

Elementary chains can be used to construct saturated structures, which are "very rich".

$\mathcal{M}$ is κ**-saturated** if for each set $X \subseteq M$ of cardinality less than κ, every set of

formulas of L_X which is finitely satisfiable in $\mathcal{M}_X$ is realized in $\mathcal{M}_X$.
$\mathcal{M}$ is **saturated** if it is $|M|$-saturated.

Theorem 2.1.9. (*Morley and Vaught*) (*i*) *If* $\mathcal{M} \equiv \mathcal{N}$, *and* $\mathcal{M}, \mathcal{N}$ *are saturated and have the same cardinality, then* $\mathcal{M} \cong \mathcal{N}$.
(*ii*) *If* $\mathcal{M}$ *is saturated,* $\mathcal{N} \equiv \mathcal{M}$, *and* $|N| \leq |M|$, *then* $\mathcal{N}$ *is elementarily embeddable in* $\mathcal{M}$.
(*iii*) *For each* κ, *every complete theory with infinite models has a* κ^+*-saturated model of cardinality* 2^κ.
(*iv*) *If* $\omega < \kappa$ *and* κ *is inaccessible, then every complete theory with infinite models has a saturated model of cardinality* κ.

At the other extreme are prime models, which are as “as small as possible”. $\mathcal{M}$ is **prime** if $\mathcal{M}$ is elementarily embeddable in every model $\mathcal{N} \equiv \mathcal{M}$. Prime models of a complete theory do not always exist, but when they exist they are unique.

Theorem 2.1.10. (*VaughtV1* (*i*) *If* $\mathcal{M} \equiv \mathcal{N}$ *and* $\mathcal{M}, \mathcal{N}$ *are prime, then* $\mathcal{M} \cong \mathcal{N}$.
(*ii*) *If a complete theory* T *has a countable saturated model, it has a prime model.*
The standard model $(\omega, +, *)$ *of arithmetic is a prime model, but it’s complete theory does not have a countable saturated model.*

A useful variant of the notion of a saturated model is that of a computably (or recursively) saturated model, introduced by Barwise and Schlipf. Assume that L and the sequence of arities of symbols of L are computable. $\mathcal{M}$ is **computably saturated** if for every n-tuple $\vec{a}$ in $\mathcal{M}$, every computable set $\Gamma(x)$ of formulas of $L_{\vec{a}}$ which is finitely satisfiable in $(\mathcal{M}, \vec{a})$ is realized in $(\mathcal{M}, \vec{a})$. While countable saturated models do not always exist, the next result shows that countable computably saturated models do always exist.

Theorem 2.1.11. (*Barwise and Schlipf*) (*i*) *Every complete theory* T *has a finite or countable computably saturated model.*
(*ii*) *Every countable structure has a countable computably saturated elementary extension.*
(*iii*) *If* $\mathcal{M}$ *is countable and computably saturated, then* $\mathcal{M}$ *is* ***resplendent****, that is, for every finite tuple* $\vec{a}$ *in* $\mathcal{M}$ *and sentence* φ *of* $L_{\vec{a}} \cup \{R\}$ *where* R *is a new predicate symbol, if* φ *is consistent with the theory of* $(\mathcal{M}, \vec{a})$ *then* φ *holds in some expansion of* $(\mathcal{M}, \vec{a})$ *to* $L_{\vec{a}} \cup \{R\}$.

Indiscernible sequences give a way to construct large models which realize only countably many types. An **indiscernible sequence** in a structure $\mathcal{M}$ is a linearly ordered set $(I, <)$ such that $I \subseteq M$, and $(\mathcal{M}, \vec{a}) \equiv (\mathcal{M}, \vec{b})$ for any two increasing tuples $\vec{a}, \vec{b}$ of the same length from I. Indiscernible sequences of n-tuples are defined in a similar way.

Theorem 2.1.12. (*Ehrenfeucht-Mostowski*) *Let* T *be a complete theory with infinite models. Then for every linearly ordered set* $(I,<)$ *there is a model* $\mathcal{M}$ *of* T *such that* $(I,<)$ *is indiscernible in* $\mathcal{M}$ *and only countably many types over* $\varnothing$ *are realized in* $\mathcal{M}$.

The ultrapower is an "algebraic" construction which produces elementary extensions realizing many types. An **ultrafilter** over a set I is a finitely additive measure on I such that every subset of I has measure 0 or 1. Given an ultrafilter U over I, the **ultrapower** $\mathcal{M}^I/U$ is formed by taking the direct power $\mathcal{M}^I$, identifying elements $f,g \in \mathcal{M}^I$ if $f(i)=g(i)\,a.e.(U)$, and stipulating that an atomic formula holds in $\mathcal{M}^I/U$ iff it holds $a.e.(U)$. In a similar way one can define the **ultraproduct** of an indexed family of different structures $\mathcal{M}_i, i \in I$.

Theorem 2.1.13. (*Łos*) *For any structure* $\mathcal{M}$ *and ultrafilter* U *over* I, *the diagonal mapping* $d(a)=I\times\{a\}$ *is an elementary embedding* $d:\mathcal{M}\prec\mathcal{M}^I/U$.

Theorem 2.1.14. (*Keisler*, *Kunen*) *There is an ultrafilter* U *over* κ *such that for every* $\mathcal{M}$, $\mathcal{M}^\kappa/U$ *is* κ^+*-saturated.*

Theorem 2.1.15. (*Keisler*, *Shelah*) $\mathcal{M}\equiv\mathcal{N}$ *if and only if there exists a set* I *and ultrafilter* U *such that* $\mathcal{M}^I/U \cong \mathcal{N}^I/U$.

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Biographical Sketch

H. Jerome Keisler is the Vilas Professor of Mathematics at the University of Wisconsin-Madison. He has worked primarily in Model Theory, and has written with C. C. Chang the classic text on the subject. He has spoken at the International Congress of Mathematicians, and in 1975 he delivered the Colloquium Lectures of the American Mathematical Society.