A BASIC EXAMPLE OF NONLINEAR EQUATIONS: THE NAVIER-STOKES EQUATIONS

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Contents

Summary

This chapter is devoted to the Navier Stokes equation as a basic example of non linear partial differential equations. The reasons of the choice of this equation are two folds. 0n one hand the mathematical study of the problem is up to now far from being completed leaving open several basic issues and results obtained up to now have involved many modern tools of mathematics. On the other hand these mathematical difficulties are closely related to the complexity of the physical phenomenon and this appear almost at all the steps of the analysis.

1. Scaling, Hierarchies and formal Derivations

Many systems of partial differential equations, linear and nonlinear, are used to describe physical phenomena such as electromagnetism, elasticity, etc… In this chapter we have chosen to describe the Navier-Stokes equations which govern the flow of a viscous fluid: they are of primary importance in fluid mechanics, and exhibit by themselves all the main features and difficulties of nonlinear equations.

There are several reasons why the study of the Navier Stokes and other closely related equations has been central in the activities of mathematicians for more than two centuries. This started probably with Euler and involved the contributions of such diverse personalities as, Leray, Kolmogorov, Arnold and others. The Navier Stokes equations are perfectly well defined mathematical objects and are paradigms of nonlinear equations. The solutions exhibit in their behavior many characteristics of genuinely nonlinear phenomena.

In view of the needs of practical applications in engineering sciences success has been limited. However the results that have been obtained contribute to our understanding of the program and this is the main idea that I would like to describe in these notes.

Fluid mechanics is in the range of our capacity of observations since the beginning of modern science. It is usual to quote some notes written by Leonardo da Vinci about turbulence in fluids. The mechanics of fluids has been used as a model for the description of phenomena that in the 18th and 19th centuries were quite mysterious, like electromagnetism (cf. Helmholtz who also made important contributions in vortex theory). Eventually the study of fluid mechanics contributed in an essential way, with the work of Boltzmann and Maxwell, to the understanding of the notion of atoms.

There is no question about the validity of the equations. Nothing has to be discovered from them concerning the intimate nature of the physics. They are just consequence of the incontestable Newton law of mechanics either applied directly to the molecules of the fluid or applied, at a more macroscopic level to elementary volumes of fluid (even if it requires some non obvious work to go from the atomic description to the continuous one). The present problems are: *how can one describe the phenomena with adequate equations, how can one compute them, and visualize the results in spite of their complexity*.

The equations involve some physical parameters and turn out to be relevant when these parameters have certain values. Therefore as an introduction it is natural to consider a "chain" of equations, hoping, as is often the case, that the next equation will become relevant when the structure of the phenomena becomes too complicated to be computed by the previous one. The Navier Stokes equations appear to be one of the main links in this chain:

I Hamiltonian system of particles,

$\Downarrow$

II Boltzmann equation,

$\Downarrow$

III Navier Stokes equations,

$\Downarrow$

IV Models of turbulence.

Each step is deduced from the previous one with the introduction of hierarchy of equations and a process of closure which in some cases leads to the appearance of irreversibility.

According to the classical Newton law, the evolution of N particles is described by a Hamiltonian system defined in the phase space $\mathbb{R}^{3N} \times \mathbb{R}^{3N}$:

$$H_N(x_1, x_2, \dots x_N, v_1, v_2, \dots, v_N) = \sum_{1\le i\le N} m\frac{|v_i|^2}{2} + \sum_{1\le i<j\le N} V(|x_i - x_j|), \qquad (1)$$

N is the Avogadro number, of the order of 10^{24}. One introduces σ, the range of action of the interacting potential V (or the diameter of the molecules when instead of (1) one uses the dynamic of elastic collisions for the evolution of the system).

To connect {I} and {II}, Boltzmann and Maxwell had the idea of studying the function $f(x,v,t)$ which describes the density of particles which have velocity v at the point x and time t; this is a solution of the so called Boltzmann equation:

$$\partial_t f + v\nabla_x f = C(f). \tag{2}$$

In (2) the left hand side represents the evolution of the function f under the sole action of the proper velocity v of the particles. The right hand side is a collision operator which models the interaction between the particles.

Formally (i.e. without proof of convergence) one proceeds as follows: first for the connection between {I} and {II} one introduce the density function

$$f_N(x_1, x_2, \ldots, x_N, v_1, v_2, \ldots, v_N)$$

which describes the probability of having at time t the first molecule at the point x_1 with velocity v_1, the second at the point x_2 with velocity v_2 and so on. This function is a solution of the Liouville equation:

$$\partial_t f_N + \{H_N, f_N\} = 0. \tag{3}$$

Assume that the particles are indistinguishable which means that at time $t = 0$ and therefore at any time t and for any permutation σ of the set $\{1, 2, \ldots, N\}$ one has:

$$\begin{aligned} &f_N(x_1, x_2, \ldots, x_N, v_1, v_2, \ldots, v_N, t) \\ &= f_N(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}, v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(N)}, t). \end{aligned} \tag{4}$$

Consider the limit of the first marginal when $N \to \infty, \sigma \to 0, N\sigma^2 \to \lambda$:

$$\begin{aligned} &f(x,v,t) = \lim f_N^1(x,v,t) \\ &= \lim \int f_N(x, x_2, \ldots, x_N, v, v_2, \ldots, v_N, t) dx_2 \ldots dx_N dv_2 \ldots dv_N . \end{aligned} \tag{5}$$

One proves that this function is a solution of the Boltzmann equation. To do so one must integrate (3) with respect to the variables $x_2, x_3, \ldots, x_N, v_2, v_3, \ldots, v_N$ and obtain, using (4), an equation of the form:

$$\partial_t f_N^1 + L_N^1 f_N^1 = M_N^1 f_N^2 \tag{6}$$

with

$$\begin{aligned} f_N^2 &= f_N^2(x,x_2,v,v_2,t) \\ &= \int f_N(x,x_2,\dots,x_N,v,v_2,\dots,v_N,t)dx_3\dots dx_N dv_3\dots dv_N \end{aligned} \tag{7}$$

and L_N^1 and M_N^2 are suitable operators.

To analyze the second marginal f_N^2 one integrates (6) and obtains an equation for the third marginal f_N^3 defined in a similar manner. Eventually one has a hierarchy (called the BBGKY hierarchy) of N equations

$$\partial_t f_N^l + L_N^l f_N^l = M_N^l f_N^{l+1},\ \ 1 \le l \le N-1,\ \ \partial_t f_N^N + \{H_N, f_N^N\} = 0 \tag{8}$$

for the marginals:

$$\begin{aligned} f_N^l &= \int f_N(x,x_2,\dots,x_N,v,v_2,\dots,v_N,t)dx_{l+1}\dots dx_N dv_{l+1}\dots dv_N,\ 1 \le l \le N-1 \\ f_N^N &= f_N\,. \end{aligned}$$

Letting N go to infinity one obtains an infinite hierarchy which is called the Boltzmann hierarchy, formally written as:

$$\partial_t f^l + L^l f^l = M^l f^{l+1}. \tag{9}$$

and one observes that if $f(x,v,t)$ is a solution of the corresponding Boltzmann equation

$$f^l(x_1,x_2,\dots,x_l,v_1,v_2,\dots,v_l,t) = \prod_1^l f(x_l,v_l,t) \tag{10}$$

produces a solution of the hierarchy (9). A uniqueness argument (of the Cauchy Kowalewskaya type) plus the fact that the initial data are assumed to be factorized leads to the conclusion that

$$f(x,v,t) = \lim f_N^1(x,v,t). \tag{11}$$

is a solution of the Boltzmann equation.

Going from {II} to {III} is simpler and probably the part of the theory which is by now the best established both at formal and rigorous levels. One observes that the collision operator of the Boltzmann equation satisfies the following invariance (inherited from the underlying Liouville equation)

$$\int_{\mathbb{R}^3_v} C(f)\Xi(v)dv = 0, \text{ for } \Xi(v) = 1,\ v_1, v_2, v_3 \text{ and } |v|^2 \tag{12}$$

and the entropy condition

$$\int_{\mathbb{R}^3_v} C(f)\log f(v)dv \le 0 \tag{13}$$

with equality if and only if f, as a function of v, is Maxwellian i.e. is given by a formula of the following type:

$$f(v) = \frac{\rho}{(2\pi\theta)^{\frac{d}{2}}} e^{-\frac{|v-u|^2}{2\theta}}. \tag{14}$$

In (14) d is the dimension of the space, ρ is the macroscopic density, u the macroscopic velocity and θ the macroscopic temperature.

The derivation of {II} from {I} corresponds to a regime where:

$$\lambda = \lim N\sigma^2, \quad 0 < \lambda < \infty. \tag{15}$$

The inverse of this number has the dimension of a length, called the mean free path or Knudsen number. On the other hand the total volume occupied by the gas is of the order of $N\sigma^3$ therefore in the above derivation this volume is very small. The term *rarefied gas* is used in this context and to go from {II} to {III} one should let $\lambda = \varepsilon^{-1}$ go to infinity. Therefore the Boltzmann equation is rescaled according to the formula:

$$\partial_t f_\varepsilon + v\nabla_x f_\varepsilon = \frac{1}{\varepsilon}C(f_\varepsilon). \tag{16}$$

The quantities

$$\begin{aligned} &\rho_f = \int f_\varepsilon(x,v,t)dv, \\ &\rho_f u_f = \int v f_\varepsilon(x,v,t)dv, \quad \rho_f\left(\frac{|u_f|}{2} + \frac{d}{2}\theta_f\right) = \int \frac{|v|^2}{2} f_\varepsilon(x,v,t)dv \end{aligned} \tag{17}$$

define the macroscopic density of momentum, the internal energy and the temperature.

At the level of equation (16) their computation would require the knowledge of higher moments, according to the formulas

$$\partial_t \int f_\varepsilon(x,v,t)dv + \nabla_x \int v f_\varepsilon(x,v,t)dv = 0, \tag{18}$$

$$\partial_t \int v f_\varepsilon(x,v,t)dv + \nabla_x \int v \otimes v f_\varepsilon(x,v,t)dv = 0, \tag{19}$$

$$\partial_t \int |v|^2 f_\varepsilon(x,v,t)dv + \nabla_x \int v |v|^2 f_\varepsilon(x,v,t)dv = 0, \tag{20}$$

and so on, as in the first derivation an infinite hierarchy of moments. However, due to the relaxation property contained in equation (13), the fact that ε goes to zero forces f_ε to become a Maxwellian and this leads to an explicit computation of the moments in term of ρ, u and θ.

In this way not only the compressible Euler equation but all the equations of macroscopic fluid dynamics for a perfect gas can be deduced, with some other convenient scaling, from the Boltzmann equation.

Much more difficult and completely unsolved questions arise for the relation between {III} and {IV}. This corresponds to situations where the macroscopic fluid becomes turbulent and when some type of averaging is necessary for quantitative or qualitative results. In spite of being the very end of the chain, this step shares in common some points with the previous one.

It is an averaging process and the “turbulent model” starts to be efficient when the original Navier Stokes are out of reach by direct numerical simulations.

In this averaging appears a hierarchy of moments which has been studied “per se” (cf. section (3)). However this is not sufficient for the following reasons:

There is up to now no well defined notion of equilibrium and relaxation to this equilibrium, with something that would play the role of the entropy as it appears in the derivation of {III} from {II} - not even an indirect proof as in the derivation of {II} from {I} by a uniqueness argument (which does not fully explain how things happen).

The parameters that would lead to turbulent phenomena are not as clearly identified as in the previous steps of the hierarchy. In some sense they are less universal and more local.

In conclusion there is up to now no case where a proof (even formal) of the validity of a derivation of {IV} from {III} is available. The arguments when they exist rely on phenomenological considerations and engineering experiments. *In spite of this lack of justification, such are the equations used to design the airplanes in which you fly!*

It is an experimental fact (*not a theorem*) *that no new mathematical results can be obtained at level* n *of the chain of equations without the knowledge of its counterpart at level* $n+1$. A tentative explanation would be that the equations at level n contains in their asymptotic behavior the properties of the equation at level $n+1$. However as said above the derivation of the model of turbulence is not for the time being accessible by first principles from the macroscopic equations and this may be a reason why theorems at the level of the macroscopic equation remain incomplete. The macroscopic equations are the cornerstone of the theory and this is the object of the next section, where comments will be made on the following issue:

1) The existence of a smooth solution of the compressible Euler equation for “short time” before the appearance of singularities due to the generation of shocks.
2) The existence of a weak solution of the incompressible Navier Stokes equation.

The derivation of {II} from {I} for the hard spheres model was fully proved by O. Lanford, but only for short times. Same is the case for the derivation of the compressible Euler equation from the Boltzmann equation (Nishida, Ukai). Eventually with the introduction of a convenient scaling one derives the Leray weak solution from the renormalized solution introduced by di Perna and Lions for the Boltzmann equation.

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Biographical Sketch

Bardos Claude was born in 1940.
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