DIFFERENTIAL EQUATIONS AND SYMPLECTIC GEOMETRY

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Keywords: Lagrangian mechanics, Hamiltonian systems, cotangent bundle, symplectic form, reduced phase space, Hamilton-Jacobi theory, Lagrange manifolds, caustics, oscillatory integrals, Maslov theory, Fourier integral operators, canonical relations, propagation of singularties.

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Summary

Already in Lagrangian mechanics, momentum and force appear as contangent vectors, not as tangent vectors. The Legendre transformation yields an equivalence between Euler-Lagrange equations and Hamiltonian systems in the cotangent bundle. The solution of a nonlinear first order partial differential equation is described in terms of a Lagrange submanifold of the cotangent bundle. In oscillatory integrals, a Lagrange submanifold describes the high frequency asymptotics via the phase functions. Fourier integral operators describe the propagation of singularities of solutions of linear partial differential equations, in terms of Hamiltonian systems on conic Lagrange submanifolds of the cotangent bundle.

1. Lagrangian Mechanics

Since Galileo and Newton, the motion of classical mechanical systems has been described as a system of second order ordinary differential equations, for the position coordinates as a function of time. More precisely, *Newton's law* states that mass time acceleration is equal to the force exerted on the system. Lagrange observed that under a general nonlinear transformation of coordinates, such as those used in the study of the perturbed Kepler problem, the acceleration transforms itself in a quite complicated way, involving among others quadratic terms in the velocities. He discovered that the equations of motion could be put into a variational form, which then automatically transforms in a much simpler way under arbitrary coordinate transformation.

In order to describe Lagrange's theory in more detail, we introduce a function $L(t, q, \dot{q})$ of time variable $t \in \mathbb{R}$, a position variable $q \in \mathbb{R}^n$, and a velocity vector $\dot{q} \in \mathbb{R}^n$. For any smooth curve $\gamma : [a, b] \to \mathbb{R}^n$, defined on a given interval $[a, b]$ on the real axis, consider the integral

$$I(\gamma) = \int_a^b L(t, \gamma(t), \gamma'(t))\,\mathrm{d}t, \tag{1}$$

in which $\gamma'(t) = \mathrm{d}\gamma(t)/\mathrm{d}t$ denotes the velocity of γ a the time *t*. If we vary $\gamma = \gamma_\epsilon$ as a function of an auxiliary parameter ϵ, differentiate the integral under the integral sign with respect to ϵ, and perform partial integration in the term in which the factor $\partial^2\gamma_\epsilon(t)/\partial\epsilon\,\partial t = \partial^2\gamma_\epsilon(t)/\partial t\,\partial\epsilon$ appears, then we obtain the variational formula

$$\frac{\partial I(\gamma_\epsilon)}{\partial\epsilon} = -\int_a^b [L]^\gamma(t)\cdot\delta(t)\,\mathrm{d}t + p(b)\cdot\delta(b) - p(a)\cdot\delta(a). \tag{2}$$

Here the dot denotes the standard inner product in $\mathbb{R}^n$, $\gamma(t) := \gamma_\epsilon(t)$, $\delta(t) := \partial\gamma_\epsilon(t)/\partial\epsilon$ is the variation of γ, the quantities $p(t)$ in the boundary terms are the momenta

$$p(t) := \frac{\partial L(t, \gamma(t), \dot{q})}{\partial\dot{q}}\Big|_{\dot{q}=\gamma'(t)}$$

(these depend on L and γ), and the quantity $[L]^\gamma(t)$ in the integrand is equal to

$$[L]^\gamma(t) := \frac{\mathrm{d}p(t)}{\mathrm{d}t} - \frac{\partial L(t, q, \gamma'(t))}{\partial q}\Big|_{q=\gamma(t)}.$$

It follows that γ is a stationary curve for $I(\gamma)$ with respect to all variations for which $p(b)\cdot\delta(b) = p(a)\cdot\delta(a)$, if and only if γ satisfies the *Euler-Lagrange equations* $[L]^\gamma(t) = 0$ for all $t \in [a, b]$.

Because the left hand side of (2) is obviously independent of the choice of the coordinate system, for any variation δ of γ, the integrand in the right hand side is invariant under coordinate transformation. Because under a change of coordinates the vector $\delta(t)$ is multiplied by the Jacobian matrix J at the point $\gamma(t)$ of the coordinate transformation, it follows that $[L]^\gamma(t)$ is multiplied by the inverse of the transposed of the matrix J. In other words, where $\delta(t)$ transforms contravariantly, the quantity $[L]^\gamma(t)$ transforms covariantly.

This allows to generalize the position space to an n-dimensional manifold Q, where $\delta(t)$ belongs to the *tangent space* T_qQ of Q at $q = \gamma(t)$, and $[L]^\gamma(t)$ is coordinate invariantly defined as a *cotangent vector*, a linear form on T_qQ. The space of all linear forms on an n-dimensional vector space E is called the *dual space of E* and denoted by E^*.E^* is a vector space of the same dimension n as E, but in view of the aforementioned different behaviors under changes of coordinates we will consistently distinguish $\mathrm{T}_q Q$ from its dual $(\mathrm{T}_q Q)^*$.

Returning to mechanics, Lagrange observed that if L is equal to the *kinetic energy* T of the mechanical system, then a computation in a rectangular inertial frame shows that $[T]^\gamma(t) =$ mass times acceleration. This led Lagrange to replace Newton's law by the equation that $[T]^\gamma(t)$ *is equal to the force f exerted on the system*. As a consequence, the force f has to be considered as a cotangent vector, a linear form on the tangent space $\mathrm{T}_{\gamma(t)}Q$, rather than as a tangent vector.

A mechanical system is called *conservative* if $T = T(q, \dot{q})$ dose not depend explicitly on the time t and if there exists a smooth function $V(q)$, called the *potential energy*, such that $f = -\mathrm{d}V(q)$. Because $-\mathrm{d}V = [V]$, it follows that in this case the equations of motion are equivalent to the Euler-Lagrange equation $[L]^\gamma(t) = 0$, with $L = T - V$. This observation too is due to Lagrange, although he did not introduce a separate notation for $T - V$, maybe because $T - V$ does not have such a clear physical meaning as the *total energy* $H = T + V$.

The coordinate invariant formulation of Lagrange turned out to be particularly useful in the search for the equations of motion in continuum mechanics, where the position of the continuum cannot be described by finitely many coordinates. However, the kinetic energy and the potential energy can be given as integrals over the continuum, and the integral (1) with $L = T - V$ is an integral over the four-dimensional space-time. This led Green (1855) and Thomson (1863) to the introduction of the equations of motion for the continuum as the Euler-Lagrange equations for the space-time integrals of $L = T - V$. Since then Euler-Lagrange equations for integrals over space-time form a basic theme in the field theories of modern theoretical physics.

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Biographical Sketch

J.J. Duistermaat was born on 20 December 1942. He received MS and PHD degrees in 1965 and 1968 respectively, both from Utrecht University. During 1969-70 he was a postdoctoral fellow at Lund University, Sweden. During 1971-72 he was a lecturer at Nijmegen University, The Netherlands and a visiting member of the Courant Institute at New York University. During 1792-74 he was a professor at Nijmegen University. He is presently a professor at Utrecht University and he took this position in 1974.