MATHEMATICAL MODELS OF CIRCULATION IN OCEANS AND SEAS

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Contents

Summary

Mathematical models of the general ocean and sea circulations are reviewed. Basic model equations traditionally called in oceanology the system of primitive equations, are considered. They follow from classical equations of the fluid dynamics of a rotating

fluid using Boussinesq, hydrostatic, incompressibility, turbulent viscosity, and diffusivity approximations.

Boundary and initial conditions are formulated and discussed together with parameterization of turbulent processes that occur on scales smaller than can be described by model calculations. Statements related to the questions of uniqueness and existence of the solutions of ocean and sea dynamics problems (for some simplifications of governing equations) are formulated.

Some conceptual alternative and generalized models of the circulations in seas and oceans are presented: shallow-water equations, ocean model in the bottom following system of coordinates, free-surface model of the sea dynamics, etc.

The problems arising in numerical calculations of sea and ocean dynamics are considered such as the choice of a differential formulation of the problem; the approximation of the differential problem with respect to spatial coordinates; the implicit methods for integrating the model in time using splitting procedures. Approaches on the basis of adjoint equations and optimal control methods are discussed for the solution of observational data assimilation and initialization problems.

1. Introduction

Life depends on climate. For many years people have been trying to understand how the climate system works and how to forecast its behavior and variability. An understanding of the climate system has a significant impact on the economic prosperity of the nations throughout the world and on ecological conditions of the Earth's changing environment.

The oceans and seas play an important role in the global climate system. Their surface and deep currents redistribute heat, salt, and chemical substances around the world. Ocean and marine thermohaline circulations have a complicated vertical and horizontal structure which is determined by atmospheric forcing, distribution of continents, sea, ice and bottom relief.

The study of the global ocean circulation has attracted considerable attention. The thermohaline component of the ocean circulation, its interaction with atmospheric and sea ice dynamics plays a key role in climatic change.

Modeling of the ocean general circulation includes a number of aspects of geophysical, mathematical and algorithmic nature, each of them being of scientific interest and thus deserving individual consideration. Two basic problems of physical and mathematical nature can be distinguished.

The first problem is associated with the studies of physical processes which form the large-scale marine and oceanic circulation and their variability. A great number of theoretical and experimental works are dedicated to this problem. The discovery and examination of the mesoscale eddies in the ocean, the key element of general circulation, is one of the most important results in this field.

The second problem involves the aspects associated with mathematical and numerical modeling of the ocean and sea dynamics. It is this problem which we discuss here.

2. Mathematical Modeling of Oceanic and Marine General Circulation

The ocean general circulation (OGC) model is based on the nonlinear equations of large-scale fluid dynamics. The model's equations follow from the Reynolds equations that describe baroclinic motion of a rotating fluid with certain approximations which are traditional in the oceanology.

The Reynolds equations are understood as equations governing the motion of an ideal fluid averaged over the time. This is associated with the fact that the motion in oceanic and sea water is virtually always turbulent. Qualitative analysis of this motion with the help of "exact" models is impossible since the latter describe the whole spectrum of turbulent pulsations, chaotic flows, regular mesoscale structures, and large-scale circulations. Therefore, when constructing mathematical models, the approach proposed by Reynolds in 1895 is employed. This approach is based on the transition from the "exact" equations to the equations that describe the turbulent motion. These equations, averaged in a special way, are called the Reynolds equations of turbulent motion.

The construction of these averaged equations is performed as follows. A desired vector-function φ with the components velocity, pressure, density, etc. is represented as a sum of rapidly and slowly varying components $\varphi', \hat{\varphi}$ respectively:

$$\varphi = \varphi' + \hat{\varphi}, \quad \hat{\varphi} = \tfrac{1}{\tau}\int_0^{\tau} \varphi \mathrm{dt} .$$

The time axis is divided into regular intervals of the length τ, and the original equations describing the nonlinear dynamics of an ideal fluid are integrated over each time interval. For a new desired vector function, we choose the slowly varying component of the old vector-function $\hat{\varphi}$ which is equal to corresponding average values on each time interval τ.

The integrals of the rapidly changing components or of the pulsations of all quantities within the intervals τ are assumed to be zero. The integrals of their nonlinear interactions are expressed through combinations of averaged functions on the basis of certain physical hypotheses of turbulent closure.

2.1. Equations of the General Circulation in Oceans and Seas

Equations of the dynamics of the seas and oceans describing an averaged large-scale evolution of turbulent thermohaline fields have the form

$$\frac{\mathrm{d}u}{\mathrm{dt}} - [1 - m(\tfrac{n}{m})' u]v + \frac{m}{\rho_0}\frac{\partial \mathrm{p}}{\partial x} = \frac{\partial}{\partial \mathrm{z}}\nu_u \frac{\partial u}{\partial \mathrm{z}} + \mathrm{F}^u , \tag{1}$$

$$\frac{\mathrm{d}v}{\mathrm{dt}}+[\mathrm{l}-m(\tfrac{n}{m})'u]u+\frac{n}{\rho_0}\frac{\partial \mathrm{p}}{\partial y}=\frac{\partial}{\partial \mathrm{z}}\nu_v\frac{\partial v}{\partial \mathrm{z}}+\mathrm{F}^v, \qquad (2)$$

$$\frac{\partial \mathrm{p}}{\partial \mathrm{z}}-g\rho=0, \qquad (3)$$

$$m[\frac{\partial u}{\partial x}+n\frac{\partial}{\partial y}(\frac{v}{m})]+\frac{\partial w}{\partial \mathrm{z}}=0, \qquad (4)$$

$$\frac{\mathrm{dT}}{\mathrm{dt}}=\frac{\partial}{\partial \mathrm{z}}\nu_{\mathrm{T}}\frac{\partial \mathrm{T}}{\partial \mathrm{z}}+\mathrm{F}^{\mathrm{T}}, \qquad (5)$$

$$\frac{\mathrm{dS}}{\mathrm{dt}}=\frac{\partial}{\partial \mathrm{z}}\nu_{\mathrm{S}}\frac{\partial \mathrm{S}}{\partial \mathrm{z}}+\mathrm{F}^{\mathrm{S}}, \qquad (6)$$

$$\rho=\rho(\mathrm{T},\mathrm{S},\mathrm{p}) \quad \text{in} \quad \mathrm{D}(x,y,\mathrm{z}) \qquad (7)$$

where

$$\frac{\mathrm{d}}{\mathrm{dt}}=\frac{\partial}{\partial \mathrm{t}}+mu\frac{\partial}{\partial x}+nv\frac{\partial}{\partial y}+w\frac{\partial}{\partial \mathrm{z}},$$

$$\mathrm{F}^*=m^2\frac{\partial}{\partial x}\mu_*\frac{\partial *}{\partial x}+mn\frac{\partial}{\partial y}\mu_*\frac{n}{m}\frac{\partial *}{\partial y}.$$

Equations (1)-(7) are written in the left-handed coordinate system x, y, z. The coordinate x is directed along the latitude (eastwards), the coordinate y is directed along the longitude (north-wards), and the coordinate z is directed downwards from the unperturbed sea surface. System (1)-(7) is considered on the time interval (0,*t*] in a three-dimensional domain *D*. The domain *D* is bounded by the boundary ∂D that consists of the unperturbed sea surface $\mathrm{z}=0$, the lateral (coastal) surface $\sum$, and the bottom relief $\mathrm{H}(x,y)$ with a normal $\mathbf{n}_{\mathrm{H}}$.

Here, u, v, w are the components of velocity vector; *T* is potential temperature; *S* is salinity; *p* is pressure; ρ is density; $\nu_u, \nu_v, \nu_{\mathrm{T}}, \nu_{\mathrm{S}}$ are the coefficients of vertical turbulent viscosity and diffusion; $\mu_u, \mu_v, \mu_{\mathrm{T}}, \mu_{\mathrm{S}}$ are the coefficients of horizontal turbulent viscosity and diffusion; *l* is the Coriolis parameter; $\mathrm{l}=2\Omega\sin y$. Quantities m and n are metric functions; in spherical coordinate system, $m=1/(\mathrm{R}\cos y), n=1/\mathrm{R}$, where *R* is the radius of Earth; Ω is the angular velocity of the Earth's rotation; in Cartesian coordinate system, $m=n=1$; the terms $\mathrm{F}^u, \mathrm{F}^v, \mathrm{F}^{\mathrm{T}}, \mathrm{F}^{\mathrm{S}}$ describe the processes of turbulent viscosity and diffusion.

Boundary conditions for Equations (1-7) can be formulated as follows.

- in the vertical coordinate, when $z = 0$

$$\nu_u \frac{\partial u}{\partial z} = -\frac{\tau_x}{\rho_0}, \quad \nu_u \frac{\partial v}{\partial z} = -\frac{\tau_y}{\rho_0}, \quad w = 0, \tag{8}$$

$$\nu_T \frac{\partial T}{\partial z} = \gamma_T (T - T_s) + Q_T, \quad \nu_S \frac{\partial S}{\partial z} = \gamma_S (S - S_s) + Q_S; \tag{9}$$

- in the vertical coordinate, when $z = H(x, y)$

$$w = m\frac{\partial H}{\partial x} u + n\frac{\partial H}{\partial y} v, \quad (\mathbf{D}u, \mathbf{n}_H) = 0, \quad (\mathbf{D}v, \mathbf{n}_H) = 0, \tag{10}$$

$$(\mathbf{D}T, \mathbf{n}_H) = 0, \qquad (\mathbf{D}S, \mathbf{n}_H) = 0; \tag{11}$$

$$(\mathbf{D}*) = (\mu_* m\frac{\partial *}{\partial x}, \mu_* n\frac{\partial *}{\partial y}, \nu_* \frac{\partial *}{\partial z});$$

- on the lateral surface $\sum$, under the assumption that the terms F^u, F^v, F^T, F^S have the form of the Laplace operators,

$$u = 0, \quad v = 0, \tag{12}$$

$$\mu_T \frac{\partial T}{\partial \mathbf{n}} = 0, \quad \mu_S \frac{\partial S}{\partial \mathbf{n}} = 0, \tag{13}$$

where $\mathbf{n}$ is normal to $\sum$.

Equations (1)-(7) are supplemented by the initial conditions for $t = 0$

$$u = u_0, \quad v = v_0, \quad T = T_0, \quad S = S_0. \tag{14}$$

The function $\rho(T, S, p)$ that defines the equation of state for sea water is chosen on the basis of empirical relations. In some cases, for example, in shallow seas, the density may depend only on temperature and salinity.

The equations are considered in the domain $D(x, y, z)$ with a fixed boundary. The domain *D* can be multiply connected because of the presence of individual islands and continents.

Traditionally, the system of Equations (1)-(7) is called in oceanology the system of “primitive” equations of the general circulation. It was obtained from classical equations of the fluid dynamics of a rotating fluid using traditional approximations: the Boussinesq approximation, hydrostatics, incompressibility, and the linear closure of turbulent exchange with momentum, heat, and salt (see *Mathematical Models in Water Sciences*).

Extension of the problem's definition: Equations of the circulation of seas and oceans (1)-(7) are formulated under the assumption that, at all moments, the fluid is stably stratified with respect to density. This physical condition requires corresponding mathematical expression, that is, the extension of the problem's mathematical statement (see *Mathematical Models in Water Sciences*). From a physical standpoint, this is parameterization of the subscale process of convective mixing. This process cannot be described by the general circulation equations and must be parameterized.

The introduction of a nonlinear dependence for the coefficient of vertical turbulent diffusion of heat and salt on the gradient of potential density along the vertical direction is one of the possible parameterizations for the convective mixing process. For example, one can set

$$\nu_{\mathrm{T}} = \nu_{\mathrm{S}} \equiv \nu_{\min} + \frac{(\nu_{\max} - \nu_{\min})}{2}(1 - \mathrm{sign}(\frac{\partial \rho_{\mathrm{pot}}}{\partial \mathrm{z}})) \qquad (15)$$

or

$$\nu_{\mathrm{T}} = \nu_{\mathrm{S}} \equiv \nu_{\min} \exp(\alpha_1 [1 - \tanh(\alpha_2 \frac{\partial \rho_{\mathrm{pot}}}{\partial \mathrm{z}})], \qquad (16)$$

where ρ_{pot} is potential density, $\nu_{\max}$, $\nu_{\min}$, α_1, α_2 are positive constants, $\nu_{\max} >> \nu_{\min}$, $\alpha_1 = \frac{1}{2}\ln[\nu_{\max}/\nu_{\min}]$.

2.2. Boundary Conditions

The general circulation problem for the seas and oceans is formulated as an initial-boundary value problem for the system of Equations (1)-(7). When setting the boundary conditions, the boundary of the domain *D* is divided into several parts: the upper interface surface the atmosphere – sea, the lower bottom surface, and the lateral coastal boundary. In most of the general circulation problems, it is assumed that all parts of the boundary are fixed.

It is assumed that the upper boundary coincides with the unperturbed sea surface $\mathrm{z} = 0$ and that the constraint $\mathrm{H}(x, y) \geq \mathrm{H}_0 > 0$ holds for the bottom surface, so that the coastal contour is also fixed. In certain cases, the upper boundary is identified as a free surface $\mathrm{z} = \zeta(x, y, \mathrm{t})$. In this case, the statement of the problems is more complicated; one of such examples of the statement is presented in what follows.

Sometimes the problem with a moving lateral boundary is also considered. However, it is not typical of the models of large-scale circulation of the seas and oceans. It can be used when examining the meso-scale dynamics of coastal zones with shallow basins.

A kinematic boundary condition for the vertical velocity at the upper unperturbed boundary $z = 0 : w = 0$ is well known as the 'rigid lid' approximation. Sometimes, a milder, linearized condition $w = \frac{\partial \zeta}{\partial t}$ is used instead. In so doing, long external gravity waves of tidal type are included into the spectrum of model solution. In the general case, one may use the exact kinematic condition with an additional condition for pressure:

$$w = \frac{d\zeta}{dt} \quad \text{at} \quad z = \zeta(x, y, t), \tag{17}$$

$$p = p_a, \tag{18}$$

where p_a is the atmospheric pressure.

The conditions for temperature and salinity at the upper boundary (9) are written in the general form. Their concrete representation is chosen depending on the situation. Starting with the Dirichlet conditions, when the asymptotics $\gamma_T, \gamma_S \to \infty$ hold, they may change up to the so-called "mixed conditions", the Newton conditions for temperature

$$\nu_T \frac{\partial T}{\partial z} = \gamma_T (T - T_s)$$

and the Neumann conditions for salinity $(\gamma_S = 0)$

$$\nu_S \frac{\partial S}{\partial z} = Q_S .$$

It seems that the use of mixed boundary conditions reflects a real situation more effectively. The heat flux into the ocean is determined by local conditions at the surface and the third kind condition (the Newton condition) holds for it. The salt flux is related with the processes of precipitation formation which may occur far from the sea surface. It is possible to turn the quantity γ_S to zero and use the Neumann condition for salinity. The salinity in this case, as well as pressure, is determined with the accuracy up to a constant, which must be taken into account when solving the problem.

2.3. Initial Conditions

The absence of data on the fields of velocity vector horizontal components u_0, v_0, temperature T_0, and salinity S_0 is the main difficulty related to the statement of the initial conditions (14). The initial fields must be set at the moment $t = 0$ at all points of the three-dimensional domain $D(x, y, z)$. When solving practical problems, the information about initial fields is extremely sparse for most of the sea and ocean basins. The measurement of hydrological fields at the same moment and at all points of the domain $D(x, y, z)$ is an almost impracticable and very expensive procedure. In this connection, important auxiliary problems arise in the dynamics of ocean: the construction of dynamically consistent fields and the construction of initial condition themselves.

The current fields dynamically correlated with the fields of temperature, salinity, and density can be used for describing the average state of a marine or oceanic medium (and as the first guess for the initial data). Observational data on temperature and salinity are available for many seas and oceans. These data, as a rule, describe their mean climatic values: yearly mean, sometimes averaged over a season and monthly mean data. The latter make it possible to restore the seasonal cycle.

Historical data arrays related to temperature and salinity represent a combination of several types of data. These are: the vertical distributions of temperature and salinity at certain points of the horizontal plane; the data on two-dimensional planes, the sections $(x, z), (y, z)$; the measurements of sea level and sea surface temperature obtained by satellite, etc. By these data, three-dimensional distributions of temperature and salinity are restored with the methods of interpolation and extrapolation. Therefore, in most cases, the fields are related to a certain time interval, say month, season, or year.

The observational data on velocity field are more scanty; they are virtually absent for many water basins. In this connection, a problem arises to construct the current fields by the measurement data on temperature and salinity.

There is the main requirement to the restored (by various methods) fields of temperature, salinity, and currents: the so-called initialization shock must be absent in the model solutions when using these data in a model as the initial conditions. The initialization shock means sharp change of a solution of mathematical model at an initial time interval of its integration. The initialization shock is caused by inconsistent initial conditions. It is accompanied by large gradients of the solution with respect to time and space. The fields of currents, temperature, and salinity that do not induce the initialization shock in a model solution can be referred to as dynamically correlated.

The problem of constructing dynamically correlated fields can be solved both by the system of equations that is further used for forecasting and by some other system. For example, one of the simplified variants can be used, that is a model which asymptotically follows from the original one.

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Bibliography

A.F.Bennett, (1992), *Inverse methods in oceanography*. – Cambridge Univ. Press, Cambridge. [This is a textbook that presents questions of mathematical modeling of ocean dynamics and some applications of inverse methods for its analysis].

A.E.Gill, (1974), *Atmosphere-ocean dynamics*.- Academic Press, New York. [This is a well known book that presents fundamental aspects of atmospheric and ocean dynamics].

Marchuk G.I., (1995), *Adjoint Equations and Analysis of Complex Systems*. – Kluwer. [This book presents essential aspects and approaches for complicated system analysisi on the base of adjoint equations].

Marchuk G.I., Dymnikov V.P., Zalesny V.B., (1987), *Mathematical models in geophysical fluid dynamics and numerical methods of their realization*. – Liningrad, Gigrometeoizdat, 296p. [The book presents and discusses methods of mathematical modeling and numerical analysis of atmospheric and oceanic general circulation].

Monin A.S., Ozmidov R.V., (1985), *Turbulence in the Ocean*. – Kluwer Academic Publishers Dordrecht / Boston / London. [This book contains general information about ocean turbulence, including observed data and physical analysis of the natural processes.].

Marchuk G.I., Agoshkov V.I., Shutyaev V.P., (1966), *Adjoint Equations and Perturbation Algorithms*, - CRC Press , Boca Raton, New York, London, Tokyo, 275 pp. [This book presents and discuss theory, methods and numerical techniques of adjoint analysis: adjoint operators in non-linear problems, solvability of adjoint equations, perturbation algorithms , applications of adjoint equations, etc.].

Washington W.M., Parksinson C.L., (1986), *An Introduction to Three-Dimensional Climate Modeling*, University Science Books, Mill Valley, California, Oxford University Press, Oxford, New York, 422 pp. [This book describes the theory of climate simulation (atmosphere – ocean – ice), including governing equations, numerical techniques, and results of numerical experiments].

Biographical Sketches

Zalesny Vladimir B.,Professor, Institute of Numerical Mathematics, Moscow, Russia, was born on 12.08.1946. and graduated from the Department of Mechanics and Mathematics of the Novosibirsk State University. He is the author of 4 books and more than 90 scientific publications in the field of mathematical modeling in Geophysical Hydrodynamics and Oceanology. His interests are in the development of numerical models of the large-scale marine dynamics, simulation of ocean general circulation and its interaction with atmospheric processes. During more than 20 years he has been teaching students at the Moscow Institute of Physics and Technology on the subject of Computational Physics.

Tamsalu Rein, Professor, Estonian Marine Institute, Tallinn, Estonia, was born on 27.03.1940 and graduated from the Department of Natural Science of Tartu University. Then he left for St. Petersburg, where he studied Physics under Professors V. Timonov and B. Kagan. In 1968 he went to Novisibirsk, Academic City, and became a scientific collaborator for Academician G. Marchuk. In 1972 he returned to Estonia and established the marine system modeling team. In the 1980's Tamsalu together with P. Malkki started a long development process through Estonian-Finnish cooperation in marine hydrodynamic-ecosystem modeling. During 1992-1999 he worked at the Finnish Institute of Marine Research. He is a specialist in the field of numerical simulation of the tidal motion, water dynamics and ecosystem modeling. He is the author of 3 books and more than 80 scientific publications.