MATHEMATICAL MODELS OF LIFE SUPPORT SYSTEMS

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**Summary**

A model is an object that is created to represent another object (entity) for some particular purpose. There are many types of models, and they range from informal mental models to formal mathematical models. The theory and practice of mathematical modeling is very important to a number of applied areas, including engineering, management, and the social sciences. Their main attractive feature is that it is generally easier to experiment with a model than with the real system being modeled. Since a mathematical model is generally not a precise representation of a real system, it is always necessary to verify and validate it. While mathematical modeling had its origin in classical mathematical physics, it is used for constructing and investigating models describing large classes of phenomena in many areas of intellectual inquiry. The basis for computer simulation is mathematical modeling.

Newton was one of the first scientists to develop the methods of mathematical modeling intensively, in particular models in physics. His works are fundamental to classical mechanics, the laws of gravitation, and theories of light. Beginning in the second half of the nineteenth century, the methods of mathematical modeling were applied successfully to develop and investigate models of phenomena in areas of physical science such as electrodynamics, acoustics, elasticity theory, hydrodynamics, and aerodynamics. Mathematical models of this class are governed mainly by partial
differential equations. While basically differential equation models, they use integral equations and integro-differential equations, variational and theoretical probabilistic methods, the potential theory, methods of the theory of functions of complex variables, and other branches of mathematics.

In connection with the vigorous development of computational mathematics, direct numerical methods using computers became especially important for the investigation of mathematical models. These are finite-difference methods and other computational algorithms for boundary value problems, which made it possible to solve new problems in gas dynamics, transfer theory, and the physics of plasma—including inverse problems in these most important directions of physical investigations—effectively using mathematical modeling methods.

Investigation using mathematical models makes it possible not only to obtain quantitative characteristics of phenomena but also to gain deep insight into the very essence of phenomena, discover hidden regularities, and reveal new effects. The tendency towards ever more detailed investigation of physical phenomena brings into being more complex mathematical models describing these phenomena, which in turn makes it possible to apply analytical methods to the investigation of these models. Mathematical models of real physical processes are, as a rule, nonlinear. In order to investigate such models in detail it is generally necessary to use numerical methods with the aid of computers.

In recent decades, mathematical models, in the form of systems of equations or relationships, have found wide application in economics, biology, environmental protection sciences, food and agriculture sciences, and other research areas.

1. Introduction

The study of mathematical models, as an element of the whole field of mathematics, is one of the most ancient sciences. A mathematical model is an approximate description of a class of real-world objects and phenomena, expressed by mathematical symbolism. Mathematical modeling is a powerful method that provides an understanding of the world and helps in forecasting and control. Analysis of a mathematical model allows us to get an insight into the phenomena under consideration. The study of a phenomenon by means of its mathematical model is designated the mathematical modeling process, or simply mathematical modeling.

The use of mathematical modeling, along with other forms of modeling, to study properties of various objects, phenomena, and processes has long been known. As early as in the middle of the fifteenth century, Leonardo da Vinci dealt with the justification of modeling methods. He tried to derive general laws and gave numerous examples. At the same time he also used analogies as modeling tools. In his works of centuries ago the actual current questions on the relation between experience and theory, the necessity to verify and generalize experimental results, and their role in cognition, were all stated. Modeling and similarity-theory related questions in the science of modeling often appeared in the sixteenth and seventeenth centuries in connection with various construction projects. The importance of these questions and their resolution—to the
construction of galleys, in particular—was pointed out by Galilei. In 1679 E. Mariott, in his treatises on colliding bodies, developed the ideas of Leonardo da Vinci and Galilei and studied the theory of mechanical similarity.

I. Newton, J. Fourier, and J. L. Bertrand developed theoretical justifications of modeling methods. They established the foundations of the modern theory of mathematical modeling and similarity theory, and formulated basic postulates regarding the experimental study of models. Since the middle of the nineteenth century successes in modeling, and mathematical modeling in particular, were connected with the development of the physical, engineering and (in recent years) economic sciences.

The role of mathematical modeling in the social and technical progress of modern society is becoming increasingly important. Mathematical modeling is directed to new applications as its methodology develops, and as the sphere of human activity is enlarged. Its forms and means of realization become increasingly diverse, and technical means of modeling are improving. Modern methods of mathematical modeling also see diverse applications: nowadays scientific investigations in natural, technical, and social sciences, engineering work (design, construction, experimental investigations, and testing for new technical systems), and engineering and management decision-making processes are all directly or indirectly connected with mathematical modeling.

In the twenty-first century the role of mathematical modeling will grow steadily. It has fundamental importance in addressing global sustainable development problems, and issues surrounding life support systems and anthropogenic impacts on the biosphere.

2. Basic Principles of Mathematical Modeling

2.1. Types of Modeling: Mathematical Modeling

Modeling methods may be classified broadly into two large groups: methods of material modeling and methods of ideal modeling. Modeling is designated as material if the connection between investigated objects and their models is objective, and if such modeling has a material character. These models are either studied by the investigator or selected from the real world. Material modeling embraces three main groups of methods: spatial, physical, and analogous modeling.

In the case of spatial modeling, models are utilized to reproduce or reflect spatial properties of the investigated object. In this case the models are geometrically similar to the investigated object. A variety of examples of graphical representations may be mentioned.

Physical modeling is intended to reproduce the dynamics of processes that happen inside the investigated object. The commonness of the processes taking place in the investigated object and the model is based on the similarity of their physical nature. This method of modeling is especially widely used in engineering, where physical modeling is applied in the design of different technical systems: for example, the investigation of aircraft forms in wind tunnels.
The third group of material modeling methods is connected with the use of material models with different physical natures, but which are described in terms of the same mathematical relations as those of the investigated object. Such modeling is designated as analogous; it is based on analogy between the mathematical description of the model and the object itself. The simplest example of analog modeling is the study of mechanical oscillations by means of an electrical system, which is governed by the same differential equations but is a more convenient medium in which to perform experiments.

Throughout material modeling, the model is a material reflection of the original object. The investigation concerns studying material influences on the object by performing experiments with the model. Hence material modeling, by its nature, is simply an experimental method.

Ideal modeling differs fundamentally from material modeling. It is based not on direct material analogy between the model and the investigated object, but on an ideal, conceptual connection between them. Methods of ideal modeling can be classified within two sub-groups: non-formalized (intuitive) and formalized.

Non-formalized modeling includes analysis of different problems when a model is not formulated: instead, a certain conceptual reflection of reality is devised and used as a basis for thinking and decision-making. Hence, any reasoning without formalized models can be considered as non-formalized modeling, the speaking, writing, or thinking individual has a certain image of the investigated object, which can be interpreted as a non-formalized model of the reality. Certain important advantages of non-formalized modeling mean that it is the main decision-making tool in the overwhelming majority of ordinary situations, and will remain so in the foreseeable future. Analysis of a situation on the basis of experience and intuition can be performed quickly, and often effectively, by the decision-making individual.

In formalized modeling, systems of signs or images serve as models; these systems are provided by rules governing their transformation and interpretation. Image modeling is a field of formalized modeling in which models are constructed from such vivid elements as elastic balls, liquid flows, and trajectories of movements of bodies. The analysis of image models is performed mentally; it can be regarded as formalized modeling if the rules governing images interactions are fixed. For instance, a collision of two molecules in an ideal gas may be modeled as an elastic interaction of balls, and the interaction result in the gas itself is conceived to be the same. Models of this type are widely used in physics in investigations that are usually termed mental experiments.

Another form of formalized modeling is sign modeling, where systems of signs are used in the capacity of models; these systems can be different: images, graphs, schemes, formulas, and so on. Mathematical modeling is the most important kind of sign modeling. In mathematical modeling the model is written in the form of a system of formulae, which are transformed on the basis of the rules of logic and mathematics. The role of mathematical modeling is enormous, both in scientific development and in practical activity: numerous examples of its importance are well known nowadays.
Example 1: Mathematical model of radioactive decay. Let the mass of a radioactive element at the time $t$ be denoted by $x(t)$, and $x(0) = x_0$ is the initial mass of the element. It is known that the rate of radioactive decay is proportional to its mass. Using this physical law we obtain the following mathematical model of radioactive decay, which consists of the following differential equation for $x(t)$ at $t > 0$ and the initial condition:

$$\frac{dx(t)}{dt} = -kx(t), \quad x(0) = x_0,$$

where $k = \text{const} > 0$ is the proportionality factor and sign “−” in the equation is taken to demonstrate the mass decrease.

The solution of this equation is given by

$$x(t) = x_0 e^{-kt}, \quad t > 0.$$

This formula allows us to establish the connection between the coefficient $k$ and the half-value period $T$: that is, the time when the half of the initial mass decays. So, from this formula it follows that $kT = \ln 2$. From this the half-value period can be found. The knowledge of the half-value period is of great importance—for example, in geology and archeology—because natural radioactive elements with known half-value periods can be used to determine the age of an object or process.

Example 2: Mathematical model of “predator–prey” system. Mathematical models are widely used nowadays to study different biological processes. Let us consider the mathematical model of the simplest of two-species system: “predator-prey”. Let the following assumptions be accepted: (1) populations of prey $N$ and predators $M$ depend on time only; (2) if species undergo no interaction then their population satisfies the Malthusian law, in other words, the rate of population change in time is proportional to current population; (3) natural death-rates of prey and birth-rates of predators are not essential; (4) the rate of prey population growth decays proportionally to the predator population, that is, to the value $cM$, $c > 0$, and the rate of predator population growth increases proportionally to the prey population, that is, to $dN$, $d > 0$.

Using these assumptions, from the relations for the rates $\Delta N/\Delta t$, $\Delta M/\Delta t$ of populations $N$, $M$ as $\Delta t \to 0$ the following Lottka-Volterra system of equations can be obtained:

$$\frac{dN}{dt} = (\alpha - cM)N, \quad \frac{dM}{dt} = (-\beta + dN)M, \quad t > 0.$$

These equations, along with values of initial populations $N(0)$ and $M(0)$, form a simple mathematical model of the system “predator–prey.” Solving these equations, it is possible to determine the populations at any time $t > 0$. 
Example 3: Equations of mathematical physics. These represent an important class of mathematical models. It includes the equations that describe mathematical models of physics and form a part of mathematics—mathematical physics. A number of these equations will be given later on.

2.2. Stages of Mathematical Modeling

The process of mathematical modeling—the study of a phenomenon using a mathematical model—can be separated into four main stages.

- The formulation of the laws that connect the main objects of the model. This stage requires a wide knowledge of the facts related to the investigated phenomena and deep understanding of their interconnections. This stage ends with the writing, in mathematical notation, of formulated qualitative notions on relations between the objects of the model.
- The investigation of mathematical problems following from creation of the mathematical model. The main problem here is to solve the direct problem: to obtain, as a result of the model’s analysis, output data (theoretical corollaries) for subsequent comparison with experimental data. At this stage mathematical tools are important for analysis of the model, and computational facilities are powerful instruments to obtain quantitative output information from solving complicated mathematical problems.
- It must be verified whether or not the model under consideration is acceptable: in other words if it provides results, within acceptable error limits, in agreement with theoretical corollaries of the model. This stage is also called as the interpretation stage of the results of the mathematical model investigation. It can also include monitoring of the model accuracy (verification of the model) on the basis of comparison of the results with other known facts, in particular with experimental data. While a model is constructed, some of its characteristics remain to be determined. The problems concerned with determining characteristics of models (parametric, functional), for example whether output information is commensurate with the observed results, are termed inverse problems. The use of practical criteria to estimate mathematical models allows us to draw conclusions regarding the correctness of the model under study.
- The final stage involves subsequent analysis of the model in light of accumulated data on the investigated phenomena themselves, enhancement of the model, or construction of a new mathematical model that is more appropriate.

In accordance with a given procedure, the process of mathematical modeling is realized in the study of many objects (processes, phenomena) and in the solution of research and practical problems.

2.3. Requirements for Mathematical Models

Let a set $S$ of properties of a real object $a$ be investigated by means of mathematical tools. Here, as before, the term object is understood in the broad sense, embracing all situations, phenomena, processes, and so on. We construct mathematical object $a' — a$ system of equations, arithmetical relations, geometrical figures, or combinations of
these notions. The study of $a'$ using mathematical tools should answer the stated questions on the properties of $S$. In these conditions $a'$ is designated as a mathematical model of object $a$ with respect to the totality $S$ of its properties. The main requirements for the chosen mathematical model (constructed or under construction) are as follows.

2.3.1. Plurality and Unity of Models

A real object can have more than one mathematical model. This is connected first of all with the necessity to investigate different systems of properties $S_1, S_2, ...$. However, even fundamentally different mathematical models of the real object are also possible in the study of the same system of properties. The choice of model type, which is essential to the direction of research, can naturally be prompted by the object itself or by reasonable precedents. However, it is worth remembering the possibility that this type may be changed. In the case of a complicated real object comparison of research, results obtained using different type models can enrich knowledge about the object, and also increase the models’ reliability.

The ability to choose a mathematical model properly from among several already known, and particularly to construct one, requires the necessary level of mathematical and special knowledge, and corresponding skills. As A. N. Tikhonov stated, “experience shows that the right choice of a model in many cases means that the problem is more than half solved.”

2.3.2. Adequacy Requirement

The most important requirement of mathematical models is their adequacy (the extent of agreement) of the investigated real object $a$ with respect to the chosen system $S$ of its properties. By this is meant:

- Correct qualitative description of the properties under consideration: for example the possibility, on the basis of model research, of drawing correct conclusions regarding the direction of changes of certain quantitative characteristics of these properties, their interconnection, the character of object’s oscillations, the stability of its state or evolution, and so on.
- The adequacy requirement also usually includes correct quantitative description of these properties with reasonable accuracy.

In accordance with whether or not condition (2) is required, we may speak about quantitative or qualitative models respectively. Instead of quantitative adequacy we also speak about the accuracy of models.

It is natural to speak not only about the adequacy of models but also about degree of adequacy. It is essential that adequacy should be considered only in terms of definite signs, which are properties accepted as basic to the research. If they are not indicated clearly then they should be implied or specified during the research.

Neglecting the fact that adequacy of mathematical models to real objects is only relative, and has its frames of usage can lead to serious mistakes, arising from ascribing...
models’ properties to real objects in an uncontrolled manner. In complicated cases non-adequacy or low adequacy of models may not be fully clear, and it is possible to speak about their adequacy with some assurance. Such assurance increases if the corollaries from the accepted model are in good agreement with firmly established facts or the results of physical experiments.

2.3.3. Requirement of Sufficient Simplicity

If we are concerned only with the adequacy requirement, then complex models are preferable to simple ones. In fact, more complicated models allow us to consider more factors which can influence the studied properties in one way or another. However, in some situations, especially unusual ones, over-complication of models can lead to large systems of equations that are difficult to investigate and solve. In practice, therefore, there is a demand for sufficiently simple models for the system of properties of interest. A model is sufficiently simple if it gives reasonably accurate quantitative or qualitative results that fulfill the stated purpose, and at the same is economical in terms of the cost of computational and other requirements of modern research effort.

The simplicity requirement to some extent opposes the adequacy requirement: as a rule the more adequate a model the less simple it is, and the more complicated is its analysis. In some cases, however, complication of a model can make its adequacy worse. This happens, for instance, if additional equations contain parameters known to be of low accuracy, or if these equations themselves are doubtful. After choosing a model we often therefore have to simplify it, and thus to pass on to a new model. More precisely, it is possible to simplify either a substantial model of the object or its mathematical model.

2.3.4. Other Requirements of Mathematical Models

The property of completeness is also essential to mathematical models. This property ensures that a model yields the best possibility, of obtaining desired outcomes using mathematical methods.

An additional important requirement of mathematical models may be designated as their productivity. It is connected with the fact that the investigated object can include different parameters (for example, masses, lengths) of its components and functional dependencies, which are supposed to be given and describe relations between values under consideration (for example, the connection between force and displacement in the case of nonlinear elasticity laws). These given parameters and dependencies, which are designated as input data of a model, have an influence on the values obtained as a result of solving the mathematical problem. The productivity requirement means that in real situations input data should actually be considered as given: in other words, it should be possible to measure them somehow, to find them in handbooks, or to estimate them in some way or another. Moreover, with regard to measurement input data should be more easily measurable than obtained data, otherwise the investigation of the model loses its sense.

The robustness requirement—for model stability with respect to errors in the input
data—is also important. We should always keep in mind that the accuracy of these data may be unknown, and such uncertainty should not influence the results of the investigation significantly. For example, instability of a mathematical model can occur because it includes some functions that oscillate rapidly, and where the values of variables are known with low accuracy.

Clearness in a mathematical model is desirable but ultimately not necessary. This property refers to a direct, clear, substantial sense of a model’s components, allowing us not only to control the model more closely but sometimes even to plan mathematical problem-solving and approximately foresee the solution, which can accelerate the procedure.

2.4. Determining Components and Relations

2.4.1. Determining Relations

The main construction elements of a mathematical model addressing a class of problems are particular constant and variable values included in the model, and the functional dependencies of some values on others. Some constant values can be given as parameters of the problem, while others are unknown; the same is true for functions. A model is constructed as if we could provide answers to specific questions if the desired unknowns are resolved.

Given and desired values and functions in mathematical models are usually connected by equations and inequalities. Moreover in many cases, especially in the problems of analysis, the model takes the form of an equation or system of equations. Even if a model contains some additional relations, usually equations constitute its essential part.

The equations involved in the mathematical model of the investigated object are written on the basis of determining relations between values, which follow from the postulates of the substantial model. Some of these postulates follow from universal physical laws like the energy conservation law, Newton’s Second law. Physical laws with restricted areas of application play similar roles; the possibility of their relevance to the problem under investigation follows from universal laws (for example, dealing with the mass conservation law in the problems of mechanics).

The number of universal and related laws is rather small, however, and therefore we also have to use other laws. Phenomenological laws like Hooke’s or Fourier’s laws, which are sufficiently well justified empirically (and, to a certain extent, theoretically) and with a restricted, empirically established area of action, are particularly widely used. When a phenomenological law is applied to the construction of a mathematical model it is important to check that its use is feasible: in other words, that the situation under investigation lies within the scope of the law’s validity. Also, the consequences of possible deviations from the law should be taken into account. Sometimes the practicality of such a use is stated in the problem conditions, but these questions will arise anyway when the results are applied to a real object.

Semi-empirical relations, obtained from a combination of qualitative arguments (in
particular dimension arguments) and either experimental results processing or other statistical treatment, have even less universal character. *Purely empiric relations* obtained by direct experiment or observation data processing are also in use. On frequent occasions they are even connected with certain measurement units.

### 2.4.2. Finite Equations

In this and the following sections of the article are described the most commonly used types of equations, appearing as components of mathematical models.

A finite equation (algebraic or transcendental), after rearrangement of all its terms to the left-hand side, takes the following general form:

\[ f(x) = 0, \]  

where \( f \) is a scalar function of scalar variable; a system of finite equations with several unknowns takes the form:

\[
\begin{align*}
&f_1(x_1, x_2, \ldots, x_n) = 0 \\
&f_2(x_1, x_2, \ldots, x_n) = 0 \\
&\vdots \\
&f_n(x_1, x_2, \ldots, x_n) = 0.
\end{align*}
\]  

For an exact solution of such a system we should check that the number of equations is equal to the number of unknowns, and these equations are independent (in other words, that none of them is obtained from others).

If all equations (2) are algebraic of the first order, we can obtain them by simple transformation in the form of a system of linear algebraic equations:

\[
\begin{align*}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = g_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = g_2 \\
&\vdots \\
&a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = g_n
\end{align*}
\]  

with the matrix \( A = \{a_{ij}\} \) of order \( n \).

### 2.4.3. Equations for Functions of One Variable

Equations more complicated than simple finite equations are also used as determining relations.

*Differential equations* are widely used in the applications of mathematics. If the unknown is a function of one variable then we use *ordinary differential equations*. A
differential equation of order \( n \) in standard form (solved with respect to the highest order derivative) takes the following general form:

\[
\frac{d^n u}{dt^n} = F\left(t, u, \frac{du}{dt}, \ldots, \frac{d^{n-1}u}{dt^{n-1}}\right),
\]

(4)

It is convenient to treat the independent variable \( t \) as time; in fact, in most applications it is really so.

General solution of equation (4) includes \( n \) arbitrary constants. So, in order to select a particular solution, \( n \) finite equations (so called additional conditions) should be introduced, which connect the values of the unknown function \( u \) and its derivatives at certain points. If we consider (4) as an equation determining the time evolution of a process, initial conditions, which fix the state of the process at the starting moment, are most often used:

\[
u = u_0, \quad \frac{du}{dt} = \left(\frac{du}{dt}\right)_0, \ldots, \frac{d^{n-1}u}{dt^{n-1}} = \left(\frac{d^{n-1}u}{dt^{n-1}}\right)_0, \quad t = t_0
\]

If the space co-ordinate is an independent variable and the unknown function is constructed on a certain interval of one-dimensional space, boundary conditions are most often used. A number \( k \) of conditions are given on the left end of the interval and \( n - k \) on the right end: usually \( n \) is even and \( k = n - k = n/2 \). Other additional conditions can be also met. A differential equation with initial (boundary) conditions occurs in the initial value problem or Cauchy problem (boundary value problem, respectively).

Systems of differential equations with several unknown functions, whose number must be equal to the number of equations, have similar form. By introducing new unknown functions every such system—and hence equation (4)—can be reduced to a system of first-order equations which have the following standard form:

\[
\frac{du_1}{dt} = f_1(t, u_1, u_2, \ldots, u_n)
\]

\[
\frac{du_2}{dt} = f_2(t, u_1, u_2, \ldots, u_n)
\]

\[
\vdots
\]

\[
\frac{du_n}{dt} = f_n(t, u_1, u_2, \ldots, u_n).
\]

(5)

To select a particular solution, \( n \) additional conditions should be also given, so initial conditions take the form:

\[
u_1 = (u_1)_0, u_2 = (u_2)_0, \ldots, u_n = (u_n)_0, \quad t = t_0.
\]

(6)
For differential equations and their systems an explicit, formula-written solution is rarely possible. However, this deficiency is compensated for by many effective methods of approximate solution, and by the availability of asymptotic and qualitative methods.

In mathematical applications and mathematical modeling, functional-differential equations (for example, differential equations with deviating variable) are widely used. As a rule, these equations are of delay or neutral type. Their simple representatives can be given by the following equations:

\[
\frac{du(t)}{dt} = f(t, u(t), u(t-h)), \quad \frac{du(t)}{dt} = f\left(t, u(t), u(t-h), \frac{du(t-h)}{dt}\right), \quad (7)
\]

where \( h > 0 \) is a given constant. Such equations appear if the simulated system has elements of delay, which lead to dependence of the evolution rate on its state not only at current time \( t \) but also at the preceding time moment \( t-h \). Functional differential equations are widely used in regulation theory, mathematical biology, medicine, economics, and other fields.

Real objects can be described by functional differential equations with even more complicated structures. In particular an equation can include not one but several discrete delays, and also “distributed delay.” This leads to integro-differential equations. In the linear case such an equation can be of the form:

\[
\frac{du(t)}{dt} = \int_{t_0}^{t} K(t,s)u(s)ds + f(t), \quad t \geq t_0, \quad (8)
\]

where given function \( K \) is termed the kernel of the equation. Integro-differential equations can have even more complicated structures.

“Purely” integral equations are also used, mostly Fredholm integral equations of the second kind:

\[
u(t) = \int_{\alpha}^{\beta} K(t,s)u(s)ds + f(t), \quad \alpha \leq t \leq \beta, \quad (9)
\]

and the Volterra equations of the second kind:

\[
u(t) = \int_{\alpha}^{t} K(t,s)u(s)ds + f(t), \quad \alpha \leq t \leq \beta. \quad (10)
\]

The corresponding first-kind equations are obtained by the replacement of the left-hand side by zero. In mathematical modeling the following eigenvalue problem often appears:
Any value of \( \lambda \) that ensures non-zero solutions of equation (11) is called an eigenvalue of the kernel \( K \). These eigenvalues determine the frequency of the so-called normal medium oscillations; the corresponding solutions determine the modes (forms) of such oscillations. They are called eigenfunctions, and geometric co-ordinates serve as independent variables. Problems of the form \( Au = \lambda u \) can be found in the simulation of different phenomena (for example, calculation of the critical state of nuclear reactor, or oscillations of elastic systems). Moreover, the form of the expression \( Au \) and of operator \( A \) can be considerably more complicated than in (11).

### 2.4.4. Equations for Functions of Several Variables

If a function of several variables is unknown then differential equation turns into partial differential equation; such equations are traditionally called the equations of mathematical physics. They appear naturally in problems connected with continuum mechanics, heat and mass transfer theory, electromagnetic theory, and other topics. Geometric co-ordinates and (for evolution problems) time usually serve as independent variables.

Partial differential equations used in engineering are separated into two classes: equations that describe a stationary state of medium, and evolution equations that describe the development of a process in a medium. Among the former, the Laplace and Poisson equations are well known: for spatial problems they take (respectively) the form:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x, y, z) \tag{12}
\]

These equations, along with their one- and two- dimensional analogues, are used to describe intense states of homogenous isotropic elastic bodies, stationary flows of ideal incompressible liquids, stationary distribution of temperature, and electric and magnetic fields. In studies of flat, homogenous plate deflection the following equations are also used:

\[
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0, \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = f(x, y). \tag{13}
\]

There are also more complicated partial differential equations and their systems. In particular, the equations become nonlinear in the analysis of large deformations, compressible flows, and other phenomena.

For stationary state equations, boundary conditions are usually additional and reflect the situation on the boundary \( \partial D \) of domain \( D \) where the solution is constructed. So for
equations (12), the values of $u$ or $\partial u/\partial n$ or their linear combination are most commonly given. These are simply boundary conditions of the first, second, and third type, respectively. Here $\partial u/\partial n$ is the derivative along the normal to $\partial D$. In the case of equations (13) boundary conditions include two equalities. For example, in the case of hard seal the values of $u$ and $\partial u/\partial n$ should be given on $\partial D$.

Among evolution equations wave equation and heat (diffusion) equation are used most often. For spatial problems they take, respectively, the form:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

where constant $a$ is equal to the speed of wave spreading for the process under consideration, and:

$$\frac{\partial u}{\partial t} = a^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

where $u$ is temperature, $t$ is time, and $a$ is the coefficient of temperature transfer. For such equations an initial condition that reflects the initial state of the process is usually given. For equation (14) it consists in providing $u$ and $\partial u/\partial t$ at the initial moment; for equation (15) only the value of $u$ at the initial moment should be given. If the situation on the boundaries of the domain is essential for the investigation, then boundary conditions should be also given: in this case the problem is referred to as an initial boundary value problem.

If recent non-stationary problems in the domains with changing boundaries are widely encountered, and the law of the boundary change is unknown, it should be defined within the solution construction. Such problems arise in studies of non-stationary movements of liquids or friable media with free surfaces, phase transition (the so called Stefan problem), and so on. In such problems an additional condition should be given, in the form of equalities, which allows us to find the boundary within other conditions. Stationary problems with unknown boundaries also exist.

It is important to point out that in recent years optimal control evolution problems have been actively solved: to do this requires additional conditions, not only at the initial time but also at the final time moment. For such problems time plays the role of an additional space co-ordinate, which multiplies the difficulties associated with their solution.

2.4.5. Extremum Problems with Finite Degrees of Freedom: Mathematical Programming

Extremum problems are widely used in mathematical modeling. These problems can be split conditionally into two classes. The first class consists of problems with finite number of degrees of freedom when the extremum point and the extremal value of the
function of finite number of variables $U = f(x)$ is unknown. The function $f$ is the criterion function (cost function), $x = (x_1, x_2, ..., x_n)$; $M$ is the domain of variables $x_i$. The second class consists of the problems involved in finding the extremum of a functional, for example:

$$f(y) = \int_{a}^{b} F(x, y(x), y'(x)) dx$$

Such a functional is designated as criterion (cost) functional where $y(x)$ is unknown function of $x$ at $a \leq x \leq b$, $y' = dy/dx$; $F$ is a given function of three variables (Section 2.4.6, below). We consider below only minimization problems, because maximization problems are reduced to them by the change-of-sign of the criterion function.

Extremum problems of functions of a finite number of variables are studied by mathematical programming. This is a field of mathematics which unifies different methods and disciplines: linear programming, nonlinear programming, dynamic programming, convex programming, and others.

The general problem of mathematical programming is to find an optimal (maximal or minimal) value of the criterion function $U = f(x)$ with respect to $x_1, ..., x_n$ in the domain of admissible values $M$ (it is written as $x \in M$):

$$U = f(x) \rightarrow \min, \quad x \in M.$$ (16)

The above-mentioned disciplines differ from each other by the use of criterion function $f(x)$ and domain $M$. For example, if $f(x)$ and $M$ are linear, we deal with the problem of linear programming; if additionally the condition of variables to be integer is imposed, we deal with a problem of integer programming; if $f$ is nonlinear, we deal with a problem of nonlinear programming.

At present in the capacity of mathematical models the problems of mathematical programming with constraints are widely used, in which the variables of the criterion function are connected by finite inequalities (free constraints). Unlike in equations of connection, the number of such inequalities may be arbitrary. These inequalities are given in the form $g(x) \leq b$, where $g = (g_1, g_2, ..., g_m)$, $g_i$ are the constraint functions, and $b = (b_1, b_2, ..., b_m)$, $b_i$ are the constraint constants. Constraints of the form $g(x) \geq b$ are certainly also possible. In addition, in the problems considered non-negativity conditions $x_i \geq 0$ may also be introduced; they are usually defined by “physical” sense.

Among the problems of mathematical programming the problems of linear programming with constraints are studied extensively and widely applied in mathematical modeling. In these problems the criterion function, and all inequalities connecting its variables, are linear. In general, the problem of linear programming is
formulated as follows. A criterion function \( U = \sum_{j=1}^{n} c_j x_j (= f(x)) \) where the constants \( c_j (j = 1, \ldots, n) \) are constrained as follows:

\[
\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, m_1 \\
\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = m_1 + 1, m_1 + 2, \ldots, m_2 \\
\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = m_2 + 1, m_2 + 2, \ldots, m
\]

or in the so-called canonical form, generalizing the above three cases, as:

\[
\sum_{j=1}^{n} a_{ij} x_j = b_j, \quad i = 1, 2, \ldots, m.
\]

A problem of linear programming takes then the following form: find non-negative numbers \( x_j \geq 0 (j = 1, 2, \ldots, n) \) that minimize the criterion function; that is, (16) holds subject to (17) or (18).

If the criterion function \( f(x) \) and the left-hand sides \( g(x) \) of free constraints are convex, but all non-free constraints are linear, a problem of form (16) under such constrains is a problem of convex programming. These problems possess an important property: they cannot have more than one solution. A problem of convex programming for which the criterion function is quadratic, \( f(x) = \sum \sum c_{ij} x_i x_j \), \( c_{ij} \) are some constants, and free constraints are linear, is called a problem of quadratic programming.

One more specific class of problems in discrete time that has found various applications in mathematical modeling in recent years is the so-called method of dynamic programming. Let the state of an object be characterized by a value \( x \) (continuous or discrete), and let this object be transferred from a given state \( x_0 \) at time \( t_0 \) to a prescribed state \( x_N \) at time \( t_N \), choosing intermediate states \( x_1, x_2, \ldots, x_{N-1} \), at time moments \( t_1, t_2, \ldots, t_{N-1} \). Let additionally the cost \( f_i(x, y) \) of the transition of the object from state \( x \) at time \( t_i \) to state \( y \) at time \( t_{i+1} \) be known. The problem is to minimize the general cost.

\[
f_0(x_0, x_1) + f_1(x_1, x_2) + \ldots + f_{N-1}(x_{N-1}, x_N) \rightarrow \min.
\]

The problems described in this section may include some random components. The value of the criterion function then becomes random, and the goal of the problem is to minimize the mathematical expectation of this value. Such problems are studied and solved by means of stochastic programming methods.
2.4.6. Extremum Problems With a Sought-For Function

A number of problems of this class is formulated as follows: for given boundary conditions

\[ y(a) = y_a, \quad y(b) = y_b \]  

and given function \( F(x, y, z) \) find a function \( y(x) \), \( x \in [a, b] \) such that

\[ f(y) = \int_a^b F(x, y(x), y'(x)) \, dx \rightarrow \min. \]  

Such problems are studied in the course of calculus of variations where different variants are considered: functional \( f \) may also include higher order derivatives; function \( y \) may be vector-valued (in other words, several functions are unknown) and depend on several variables; boundary conditions may not be given on the whole boundary or even not given at all, and so on. Conditional extremum problems are also considered: for instance, if in problem (19), (20) one or several additional conditions of the following form are given:

\[ g(y) = \int_a^b G(x, y(x), y'(x)) \, dx = 0. \]

Such a variational problem is designated an isoperimetric problem.

If function \( F \) is defined for all values of the variables and has continuous derivatives the function \( y \), which yields the extremal value of the functional, is a stationary point of the functional \( f \). From the stationary condition the so-called Euler equation, which must be satisfied by the unknown function \( y \), can easily be derived. So, for the functional (20) the Euler equation takes the form:

\[ F_y' - \frac{d}{dx} F_y'' = 0. \]

Thus with (19) we obtain a boundary value problem for the second-order differential equation, which yields the unknown solution \( y(x) \). One more class of extremal problems with unknown function is the class of optimal control problems. One possible general form of such problem is as follows. A system of differential equations:

\[ x'_1(t) = g_1(t, x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_m(t)) \]
\[ x'_2(t) = g_2(t, x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_m(t)) \]
...........................
\[ x'_n(t) = g_n(t, x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_m(t)) \]  

\[ (21) \]
and initial and boundary conditions:

\[ x_1(t_0) = x_{10}, x_2(t_0) = x_{20}, \ldots, x_n(t_0) = x_{n0} \]
\[ x_1(T) = x_{1T}, x_2(T) = x_{2T}, \ldots, x_n(T) = x_{nT}. \]  

are given. Here functions \( x_1, x_2, \ldots, x_n \) describe the evolution of the controlled system while functions \( u_1, u_2, \ldots, u_m \) are the control set limited by given restrictions, for example of the form:

\[ |u_i(t)| \leq u_{i0} \quad (i = 1, 2, \ldots, m) \]  

The optimal control problem has the following form: find a control to minimize given criterion functional

\[ f(x, u) = \int_{t_0}^{T} \varphi(t, x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t)) dt \rightarrow \min \]  

provided that conditions (21)–(23) are satisfied. If this problem has solution \( (y, u) \), the control \( u(t) \) in this pair is termed optimal.

2.5. Classification of Mathematical Models

Mathematical models may be classified in different ways according to the area of application, the characteristics of simulated objects, the degree of the model’s detail, and other criteria.

2.5.1. Structural and Functional Models

Usually a mathematical model reflects the structure of a simulated object and essential properties and interconnections of the object’s components; such a model is termed structural. If a model reflects only the result of the object’s functioning (for example, its reaction to external impacts) it is called a functional model or a black box model. Combined models are also possible.

2.5.2. Discrete and Continuous Models

There are quantities of two types: discrete (which take separated values allowing natural numeration) and continuous (taking all the values of an interval). The combined case is also possible when a quantity acts on one interval as discrete and on another as continuous.

Similarly, both substantial and mathematical models may be discrete or continuous, or combined. There is no barrier of principle between these types. In specification or modification of a model the discrete case can be converted to continuous and vice versa;
the same can happen while a mathematical problem is being solved.

2.5.3. Linear and Nonlinear Models

Linear dependence of one variable on another refers to the proportionality of their increments: that is, the dependence of the form \( y = ax + b \), which yields \( \Delta y = a\Delta x \) (\( \Delta \) denotes increment). Similarly, the linear dependence of three variables is dependence like \( z = ax + by + c \), whence \( \Delta z = a\Delta x + b\Delta y \). Hooke’s law, Ohm’s law, and the Thermal Expansion law are typical examples of linear dependence. In reality these dependencies are only approximately linear, but very often the linearity assumption provides adequate results in practice.

The notion of a linear model is similarly defined. It is applied to models of the objects that are considered as transformers, for which every input provides a certain output. If, for example, we study the problem of a straight rod deflection, its density may be treated as an input and the deflection as an output. In mathematics such a transformer is designated as an operator.

We assume that starting points of the input and the output are chosen so that zero output corresponds to zero input. Then the model is called linear if it satisfies the superposition principle; that is, the output due to all the inputs acting together on the model happens to be the sum of the individual outputs due to each input acting alone on the model, and the output is multiplied by a certain value if the same happens with the input. If this principle fails, the model is nonlinear. Linear models are usually described by linear non-homogenous equations (algebraic, differential, or others), where the non-homogenous term corresponds to the input and the solution to the output.

The linearity of a model simplifies drastically the construction and the analysis of the solution of the given mathematical problem. But there are also nonlinear objects (including phenomena) when linear models fail to portray reality fully. The advantages of linearity are so great, however, that replacement of nonlinear relations by linear approximations—in other words, the linearization of relations and models—is quite common. Such a linearization is usually performed in two cases: either if the experiment demonstrates only small deviations from linear dependence (like for Hooke’s law) for the considered domain of variables, or if this domain is small and we replace increments by their differentials dropping the smaller terms. In the latter case linear interpolation can be also applied.

2.5.4. Deterministic and Probabilistic Models; Other Types of Model

Mathematical models can include random components: scalar or vector variates, random sequences or functions, random structures, which satisfy random laws. Such models are designated probabilistic or stochastic models, and are unlike deterministic models which contain no such components. Probabilistic models are studied using the methods of probability theory.

Other characteristics are also used to classify models. Static and dynamic (evolution) models are essentially different. Dynamic models study changes of the considered
object in time. In between there are quasi-static, stationary, and quasi-stationary models. In quasi-stationary models it is assumed that the changes occur so slowly that at every time moment it is possible to consider the object as static, and to treat time as an additional parameter. In stationary models it is assumed that processes happen but that the object undergoes no change over time. Quasi-stationary models can similarly be defined.

In connection with the above-outlined types of models the terms used also deserve mention: stationary or periodic process is usually designated steady; the process of transition from one static state or steady process to another is termed transitional.

2.5.5. Classification of Mathematical Models of Earth’s Life Support Systems

Earth’s life support system includes several important elements: the hydrosphere (water), atmosphere (air), lithosphere (the earth’s crust and upper mantle), the earth’s crust (soil and rock), biosphere (both living and lifeless organisms), mantle (partially molten rock), and the earth’s core (liquid). Human beings are a part of the biosphere: the totality of living organisms and non-living matter concentrated in surface layers of the atmosphere, in the hydrosphere, and in the upper layers of the lithosphere. This totality of living and non-living components, which interact with each other and their inorganic habitat (energy and chemical substances) on a planetary scale, is designated the ecosphere. The sun is the main energy source for the ecosphere.

A natural desire of human beings is that the earth’s life support systems, and in particular the ecosphere, continue to function stably. Over the last three decades, however, human activities that upset natural processes of solar energy transformation in the ecosphere, and its return to cosmic space in the form of heat radiation, have been causing much concern. As a result of these activities, global climatic characteristics are changing. This problem, along with other problems generated by modern society (population, poverty, problems with freshwater and natural resource storage, environmental pollution), constitutes a key general challenge: that of achieving both sustainable development and security for life on earth. All the above-mentioned problems are global in character. Addressing them rests heavily upon mathematical modeling and mathematical models, which are often the only practical tools to use in their study.

The problems of life support systems are investigated by many sciences. In this situation it is impossible to provide a complete classification of the mathematical models used for investigating life support systems and considering the challenge of sustainable development. Only possible approaches to such a classification may be proposed. One of them is presented below.

Let us consider only “global” processes in the biosphere where the main forms of life thrive. It is also assumed that the influence of the lithosphere on these processes can be taken into account by means of its action on the processes in the atmosphere and the oceans (influence on atmospheric processes, of ocean bottom relief on water circulation, and others). The state of the upper soil layer can be considered in connection with agricultural problems. Let the cryosphere also be considered as part of the hydrosphere.
Under these assumptions, the following life support systems on the earth can be selected:

- **Atmosphere** and its chemicals, general atmosphere circulation, short-term troposphere changes (temperature, air pressure, humidity, precipitation activity, solar radiation) termed **weather**, and also **climate** (general type of atmosphere and weather conditions in a certain region over an extended time period: as a rule, not less than 30 years).

- **Hydrosphere** and its chemicals, water circulation in the world ocean, its ecosystems, and general cyclic water flow in nature.

- **Energy** as one of the basic components of the interconnection of communities with the inorganic environment, its sources, methods of transformation and transmission.

- **Nutritious substances** and foodstuff resources (including soil and agricultural resources, water and forest resources, state of soil and forests).

- **Living organisms**, populations, species, communities, surface ecosystems.

- **System of direct and inverse interconnections** of communities’ vital functions with basic components of ecosystems, the influence of anthropogenic factors on the surrounding environment (including various components of atmosphere and hydrosphere pollution, climate change, reduction and extinction of species, soil pollution, and poisoning).

Turning now to models appropriate to studying the above-listed components, it is possible to select the following classes of mathematical models representing earth’s life support systems:

- **models of water**: its sources, movement of water masses in the world ocean, global water circulation, water pollution, ecosystems of the hydrosphere
- **models of energy**: including solar energy, energy sources, methods of transformation and transmission
- **models of atmosphere and climate**: including models of atmospheric processes and weather forecasting, atmosphere chemical composition change, aerosol distribution, climate changes and robustness
- **models in food and agricultural sciences**: including problems connected with soil, its pollution and erosion
- **models connected with the existence of organisms, species, populations, and communities**: that is, **mathematical models in biological sciences**, and also **mathematical models in health and medical sciences**
- **models of human societal relations and global biosphere processes**: including models of social, political, and economical interrelations in human society, demographic models, models of anthropogenic impacts on the biosphere, and models of global processes of the development on the earth.

We may remark that these classes of models can intersect, and that some mathematical models can be included within different classes at the same time. For example, models of the energy balance on the earth may be included both in the class of energy models and in the class of biosphere models, as well as among models of human societal development. This circumstance emphasizes the conditional character of any allocation to the classes above in respect of the earth’s life support systems, and of the problems of
sustainable development.

2.6. General Methods of Analysis; Simplification and Specification of Models

One general method employed in the analysis of models is the *dimension analysis* of values including mathematical models.

2.6.1. Dimension Analysis

*Dimension analysis* is the method of establishing a connection between physical values which are essential to the phenomenon under investigation; this method is based on the treatment of dimensions of these values. In dimension analysis the problems of different systems of units, the questions about the choice of primary values and corresponding experimental units, and the formation of secondary units for the values which are determined through primary units, are all considered.

Some of the characteristics of simulated objects are measured in certain units which make direct (mechanical, physical, or economical) sense; for example mass in grams, temperature in Kelvin grades, or Gross National Product in a currency. Such values are termed *dimensional*, their numerical value depending on the choice of units. Among them values with independent (basic) dimension, or *dimensionally independent* (primary) values, are selected. For example, if the CGS unit system (centimeter, gram, second) is used to describe mechanical phenomena, the dimensions of length $x$, mass $m$, and time $t$ are independent and cannot be expressed in terms of the others.

However, the dimension of the kinetic energy $E = \frac{mv^2}{2}$ is expressed in the dimensions of basic values (length, mass, and time) as $[E] = [m][x]^2[t]^2 = g\text{cm}^2\text{s}^{-2}$, which is designated as the *dimension formula*. Such values are termed *dimensionally dependent* (secondary). Here the dimension of the value $f$ is denoted by $[f]$. It is worth pointing out that phenomena and processes can be described by dimensionless values, too.

Different units may be considered as primary units. Secondary units are expressed via primary units by *dimension formulas*, which enable us to determine numerical scale multipliers in order to convert corresponding characteristics while primary units are changed.

If the numerical value of a quantity is independent of the choice of scales for primary units then such a quantity is *dimensionless*, or abstract. Reynolds number $Re = \frac{\rho v l}{\mu}$, Froude number $Fr = \frac{v^2}{\sqrt{gl}}$, Mach number $M = \frac{v}{a}$, and cavitation number $\kappa = \frac{2\Delta P}{\rho v^2 l^2}$ are all examples. Here $\rho$ — density, $v$ — velocity, $l$ — length, $\mu$ — dynamic viscosity coefficient, $\Delta P$ — characteristic difference of pressures, and $g$ — acceleration due to gravity.

The analysis of dimensions within the procedure of *de-dimensioning* (de-scaling) and of the transition to dimensionless quantities are always useful in the investigation of...
mathematical models, because they can provide important preliminary information on the object. The analysis of dimensions is directly connected with the notion of physical similarity, which is studied in the similarity theory.

2.6.2. Similarity of Objects

Similarity theory is the doctrine of the study of objects based on the notion of their similarity. Two objects are said to be similar if it is possible to obtain quantitative characteristics of one object using the quantitative characteristics of the other. These characteristics can be obtained by simple calculations, which are analogous to a transition from one system of units to another. For any totality of similar phenomena, all corresponding dimensionless characteristics (dimensionless combinations of dimensional quantities) have the same numerical value. The converse is also true: if all dimensionless characteristics for two phenomena are the same, then the phenomena are physically similar. Dimensional analysis and similarity theory are closely interconnected, and are well established in the foundations of experiments with models. In such experiments a real phenomenal investigation is replaced by the study of an analogous phenomenon using reduced-scale or enlarged-scale model.

Let a phenomenon be determined by \( n \) independent parameters (including dimensionless ones). Let the dimensions of the determining variables and physical constants be expressed via dimensions of \( k \) of these parameters with independent dimensions \( (k \leq n) \). Then it is possible to construct only \( n-k \) independent dimensionless combinations, and all sought dimensionless characteristics of the phenomenon can be considered as functions of these \( n-k \) quantities. Among all dimensionless quantities constructed from the determining characteristics, a certain base can be always indicated, which is a system of dimensionless quantities that determine all others.

Dimensional analysis and similarity theory constitute the basis for many approaches to obtaining mathematical models, using them, and simplifying them.

2.6.3. Methods of Simplifying and Specifying Models

Quite often it is possible, after construction of a complicated mathematical model, to simplify it in one way or another, passing on to a new, simpler (and usually approximate and less adequate) model. This simplified model can be sufficient for the purposes of the investigation; otherwise the results of its analysis can be applied to the study of more complicated models. Sometimes from the outset we construct an approximate model, keeping in mind the needs of its future specification. Therefore the following methods of simplification or specification play an essential role in mathematical modeling:

- working assumptions and simplification of models
- simplification of equations
- averaging-out of fast-oscillating entry dependencies
- small parameter method
- usage of experimental data and the “data assimilation” procedure.
The small parameter method (or perturbation method) is widely used in applied mathematics, both for simplifying mathematical models and specifying the solution obtained from the simplified model, and for clearing up the errors arising from this solution.

There are two types of problem when the small parameter method is applied. In the first of these the small parameter is already introduced in their statement: the goal of investigations is to clarify the influence of this parameter on the solution. The method yields asymptotic formulae, which demonstrate such an influence. The second type have no small parameter in their statement, and such a parameter $\varepsilon$ should be introduced to apply the method. Let $\varepsilon$ be such that at $\varepsilon = 0$ we have the modified (so-called non-perturbed) problem, and at some $\varepsilon = \varepsilon_0$—the original problem. Then the solution of the problem with $\varepsilon$ should be expanded in powers of $\varepsilon$, putting $\varepsilon = \varepsilon_0$.

At present perturbation methods are widely used to investigate and solve numerically different applied problems. One of the most general indicators for classifying perturbation methods is whether the studied perturbed problem is singularly perturbed or regularly perturbed with respect to the simplified problem. Singularly perturbed problems are those where the perturbation operator constitutes the principal part of the original complicated problem. (By an operator is meant here a system of relations which actually forms the mathematical model or the problem.) Problems for differential equations with a small parameter as the highest derivative form a large group of this class. In regularly perturbed problems, the perturbation operator has a subordinate character with respect to the principal part of the operator of the problem. If the operator of the perturbed problem is an analytic function of some parameter, such a problem is termed a problem with analytic perturbation.

The essence of regular perturbation algorithms is as follows. Let us assume that the determining relations of a linear mathematical model are written in the form of an operator equation:

$$ Au = f, \quad (25) $$

where $f$ is the input data function (vector-function), which is assumed to be known, and $u$ is the solution (that is, the output data of the model). $A$ is a linear operator, which determines rules (determining relations, equations) for given function $u$ to obtain function $f$. Equation (25) is often said to be non-perturbed basic equation. Along with (25) the perturbed equation:

$$ A_\varepsilon u_\varepsilon = f_\varepsilon \quad (26) $$

is introduced, where $A_\varepsilon = A + \varepsilon \cdot \delta A$; $\delta A$ is the perturbing operator; $\varepsilon$ is a numerical parameter: $f_\varepsilon = f + \varepsilon \cdot \delta f$. The formal scheme of the perturbation algorithm for (26) is as follows. We assume that there exists a solution $u_\varepsilon$ for equation (26), which can be represented in the form of series in powers of $\varepsilon$: 
\[ u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \]  \hspace{1cm} (27)

Substituting (27) into (26), and equating the coefficients at the same powers of \( \varepsilon \), we arrive at the following equations for \( u_i \), \( i = 0, 1, 2, \ldots \):

\[
\begin{align*}
Au_0 &= f \\
Au_1 &= \delta f - \delta Au_0 \\
Au_i &= -\delta Au_{i-1}, \quad i = 2, 3, \ldots
\end{align*}
\]  \hspace{1cm} (28)

If these equations are solvable and functions \( u_i \) are found then, using (27), it is possible to construct the function \( u_\varepsilon \), which is referred to as a formal solution of equation (26).

The function:

\[ u_\varepsilon^{(N)} = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots + \varepsilon^N u_N \]  \hspace{1cm} (29)

is referred as \( N \)-th order approximation to \( u_\varepsilon \).

Let us consider the fact that, to find functions \( u_i \), the equations from (28) with the same operator \( A \) of non-perturbed problem (25) have to be solved. Usually this operator is simpler than the given more complicated perturbed operator \( A_\varepsilon \). This fact illustrates the main reason for the wide use of the perturbation algorithms.

The described perturbation algorithm can be used both for the simplified model and for specifying the solution obtained from the simplified model. Sometimes the transition from complicated model (26) to system (28) and the search for the solution of only one simplified equation \( Au_0 = f \) enables us to find an acceptable approximation \( u_0 \) to the desired characteristics of complicated model (26). To improve the accuracy of the solution the correcting functions \( u_1, u_2, \ldots \) can be found.

Perturbation algorithms have found wide use in the study of nonlinear mathematical models, where they are among the most favored basic methods for both investigation and approximate solution. Singularly perturbed models are also investigated using these methods, although in such cases the algorithms are more complicated.

In conclusion it is worth noting that simplified mathematical models and simplified formulas possess a number of obvious advantages. However, we often do not know whether a simplified model or formula can be applied in a specific case. To clarify the purview of a simplified method usage, a comparison of the obtained solution with a more exact one should be performed. Then, subject to the results of this comparison, an additional specification for the model used can be made. One such means of specifying the model is the attraction of experimental data and the use of a special mathematical procedure: the solution of the data assimilation problem.

The essence of the data assimilation problem is as follows. Let a mathematical model be
written in the form:

\[ Au = f \] (30)

where \( f(x) \) is a function (vector-function) of input data of the model, \( u(x) \) is the sought-for solution (the characteristic of the object), \( A \) is the operator of the problem (the set of determining relations), and \( x = (x_1, x_2, \ldots, x_n) \) is the set of all independent variables of the problem. Let us assume that \( f \) is presented in the form \( f = f_0 + v \) where \( f_0(x) \) is a known function, and \( v(x) \) is an unknown function of data which are either unknown or represent the sought corrections to \( f_0 \).

At the same time let a number of observation data (experimental data) for the simulation object (process, phenomenon) be given, and the totality of these data be denoted as \( \phi_{\text{obs}} = (\phi_{\text{obs},1}, \phi_{\text{obs},2}, \ldots, \phi_{\text{obs},N}) \). Let us assume that these observation data are related to the sought characteristic \( u \) by the equation:

\[ Cu = \phi_{\text{obs}} \] (31)

where \( C \) is the observation operator: the set of relations and operations over \( u \), which yield \( \phi_{\text{obs}}, \; Cu = (Cu_1, \ldots, Cu_N) \). The exact equality \( Cu = \phi_{\text{obs}} \) holds for the adequate model only. Therefore to reach this equality, or to approximate it properly, is simply the goal of the stated problem, which can be formulated as follows: find a characteristic \( u \) and a function \( v \) such that (30), (31) hold. This problem is in fact the inverse problem when a set of parameters of the model \( v \) is sought for within \( u(x) \). If the problem (30), (31) is solved the vector \( f = f_0 + v \) is constructed, and the model is the most adequate to the real process. It is very rare that problem (30), (31) can be solved exactly, however. Therefore the following method of transforming the data assimilation problem to optimal control problem is widely used. The criterion function (cost functional) is constructed on the basis of \( \phi_{\text{obs}} \) and relation (31). For example, the criterion function can take the form:

\[ J_\alpha(u, v, \phi_{\text{obs}}) = \alpha \int_\Omega |v|^2 \, dx + \sum_{i=1}^N \left| (Cu)_i - \phi_{\text{obs},i} \right|^2, \] (32)

where \( \alpha \geq 0 \) is a numerical parameter (regularization parameter), and \( \Omega \) is the domain of variables \( x_1, x_2, \ldots, x_n \). Now the data assimilation problem is formulated as follows: find \( u(x), \; v(x) \) such that

\[ Au = f_0 + v, \; J_\alpha(u, v, \phi_{\text{obs}}) \rightarrow \inf_v. \] (33)
This is the problem of finding the optimal function $v$, which minimizes $J_\alpha$, or just the optimal control problem. At $\alpha > 0$ this problem has usually a solution, which can be constructed by different methods of extremum problems solution, optimal control theory, and inverse problems theory. Problem (33) is also said to be a variational data assimilation problem. The convergence of the solution of (33) to the solution of (30), (31), as $\alpha \to 0$ is one of the most important problems, because the assumption of the adequacy of the model (30), (31) to the simulated phenomenon is stated. This problem is solved by the use of various results from different areas of mathematics.

3. Mathematical Models in Water Sciences

The role of water in the evolution of the biosphere and life support on earth is fundamental. Water resources, as the most important biosphere component, determine the structure and functioning of ecosystems, global processes of natural matter, and energy transfer and circulation. While water reserves on the earth may be considered abundant, 97 percent of their volume is liquid salt water in the oceans and seas, 2 percent is within polar ice and glaciers, and only 1 percent is circulating freshwater and water vapor. It is on this 1 percent that life processes on earth depend. There are many scientific and technological problems in understanding and managing the world’s freshwater supply. The key role in addressing most of them belongs to mathematical modeling.

Mathematical modeling in water sciences describes a wide spectrum of processes, with different time and space scales, in various water bodies on the earth: circulation of water in artificial reservoirs in laboratories and engineering (industrial) structures; currents and waves in rivers, lakes, and coastal areas; water circulation in the sea and ocean; and wave motions are all referred to here. The physical processes described by mathematical models are of complex character. From the mathematical point of view the main distinctive feature of models concerning water sciences is their non-linearity. This makes it difficult—and in most cases impossible—to find exact solutions, and the need to use numerical calculations and experiments arises.

The historical development of water sciences is primarily connected with the development of hydrodynamics, one of the oldest sciences. Hydrodynamics deals with the laws of fluid motion, and physical objects with fluidity as their main property. Fluidity is an ability to deform under the influence of weak outside forces: it is also typical of gases, explaining why they are often referred to as fluids. The main difference between gases and fluids (liquids) is that they have varying degrees of compressibility. A fluid such as water, for instance, is slightly compressible. Its ability to form a free surface is connected with this property of slight compressibility if it is contained in a reservoir.

In the description of water movement mathematical models in hydrodynamics are used on the basis of general laws of mechanics, particularly the conservation law dealing with momentum, mass, and energy. In addition, models require a state equation relating several of the components in the other equations, and sometimes additional equations also: for example, a salinity equation for seawater. They are combined with some conditions and parameters to characterize the specific properties of water environment:
Applying the general laws of mechanics, and considering supplementary physical dependencies and parameters, the problem of studying water motion is reduced to some systems of differential or integro-differential equations. As the movement described is restricted by boundaries such as lateral walls, beds, and free surfaces, great attention is paid to the formulation of boundary conditions when constructing mathematical models. Boundary conditions are formulated on solid motionless and moving parts of the contour lines, liquid, and free boundaries. The system of equations and boundary conditions also includes a description of the influence of the external forces involved: for example the influence of the wind on the sea surface. Initial conditions are added to the systems to characterize the state variables at the initial moment (for instance, motionlessness).

With the development of theoretical methods and growth in practical experience, nonlinear models are increasingly used in hydrodynamics, and special parameterizations of turbulent processes are taken into account. As the models become more complex, finding exact solutions becomes impossible in most cases. This is particularly so in geophysical hydrodynamics, a young science which developed out of classical hydrodynamics in the last quarter of the twentieth century. Geophysical hydrodynamics studies geophysical hydrodynamic objects and phenomena: turbulent flows stratified by density on the rotating earth. The main peculiarities of physical problems and mathematical models in this field are rotation, stratification, the complicated geometry of the water bodies, and the turbulent character of water flow. Because of this complicated mathematical and physical character, studies in geophysical hydrodynamics are connected to, and determined by, the development of approximate methods, most importantly in numerical modeling.

Rapid progress in the solution of hydrodynamics and geophysical hydrodynamics problems has been linked with the rapid development of computers at the end of the twentieth century, and with developments in computational mathematics, optimal control, statistical methods, and data processing techniques.

3.1. Some classes of mathematical models in water sciences

The variety of mathematical models in water sciences is very large. Considering those which are directly connected with earth’s life support systems, the following classes of mathematical models can be selected on the basis of the hydrological cycle presented in Figure 1.

- models of water circulation in oceans and seas; of waves (tidal waves, wind waves, internal waves, and tsunamis); in coastal zones and estuaries
- models of flows in rivers and lakes; in hydro-geological processes (infiltration, flow processes in groundwater, inland seas and lakes, around dams, and in reservoirs); of snow cover, glaciers, ice sheets, and ground ice
- models of extreme events (floods, soil erosion, avalanches, mudflows, landslides); of evaporation, clouds, condensation, and rain; of desalination processes
- models in hydrosphere monitoring and data assimilation processes; in water
resources planning and management of global water circulation and global reserves of fresh water; of sustainable development problems.

Mathematical models of classical hydrodynamics and geophysical hydrodynamics constitute the theoretical basis for many of the above classes of models. Depending on the problem(s) of life support systems under investigation, the list of classes of mathematical models may be modified.

Figure 1. Hydrological cycle

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Valery Agoshkov, Doctor of Sciences, is a Professor of Mathematics and a Leading Researcher of the Institute of Numerical Mathematics, Russian Academy of Sciences. From 1970 until 1980 he was a researcher at the Computing Center of the Siberian Division of the USSR Academy of Sciences in Novosibirsk. In 1975 he received his Ph.D. on the theme “Variational Methods for Neutron Transport Problems” in the field of numerical mathematics. From 1981 to date he has been working at the Institute of Numerical Mathematics, Russian Academy of Sciences (Moscow). He is the head of the Adjoint Equations and Perturbation Theory Group of the Institute of Numerical Mathematics. In 1987 he defended a Doctoral thesis on the theme “Functional Spaces, Generalized Solutions of Transport Equations and their Regularity Properties” in the field of differential equations.

His research interests include the principles of construction of adjoint operators in non-linear problems; the solvability of equations with adjoint operators; domain decomposition methods and the Poincare-Steklov operator theory; numerical methods for partial differential equations; functional spaces, boundary value problems for transport equations and regularity of solutions; inverse problems; optimal control theory and its applications in the data assimilation processes. His scientific results in the above fields have been published in more than 160 papers and nine books.