CHAOS AND CELLULAR AUTOMATA

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Contents

Summary

In this chapter, we are concerned with Chaos theory and Cellular Automata theory. These two theories have in common the fact that they were very popular some time ago and the fact that they are intimately connected to the development of digital computing.

The chapter is divided in two parts the first one developed to chaos, the second one devoted to cellular automata. We do not try to make a survey of all aspects of these theories. We just concentrate on few aspects which seem to be relevant to modeling and give cautionary advises relative to these theories in the process of modeling.

1. Chaos

1.1. Introduction

Compare the two signals in Figure 1a:

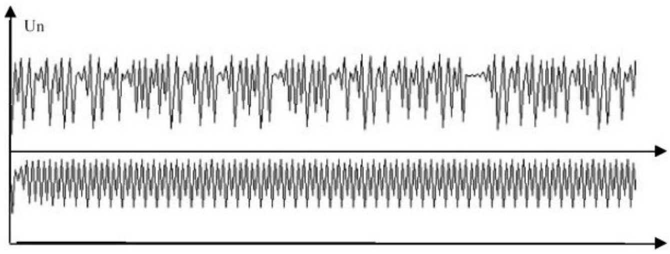

Figure 1a: Two signals

They look very different; the one below is nicely periodic since the one above has apparently no regularity. It looks random. But it is not ! Both signals are obtained by iteration of the very simple scheme:

(1) $U_{n+1} = \lambda U_n (1 - U_n)$

the first one with $\lambda = 3.8$ the second one with $\lambda = 3.5$. This is a definitely deterministic process ! The first signal is called *chaotic* because it is different from a pure *random* signal.

To make the difference between chaos and pure random we compare the chaotic signal above with the signal obtained by plotting successively the result of a call to the "random" function of a computer. One obtains the function shown in Figure 1b:

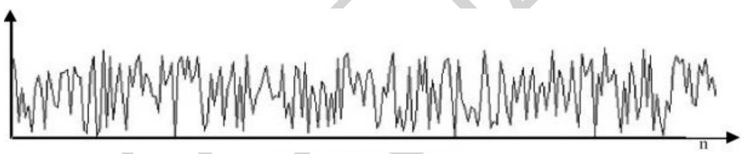

Figure 1b: A random function

This signal looks similar to the chaotic one but if we plot, not the value of the signal against time, but the value at time $n+1$ against the value at time n we obtain the function in Figure 2:

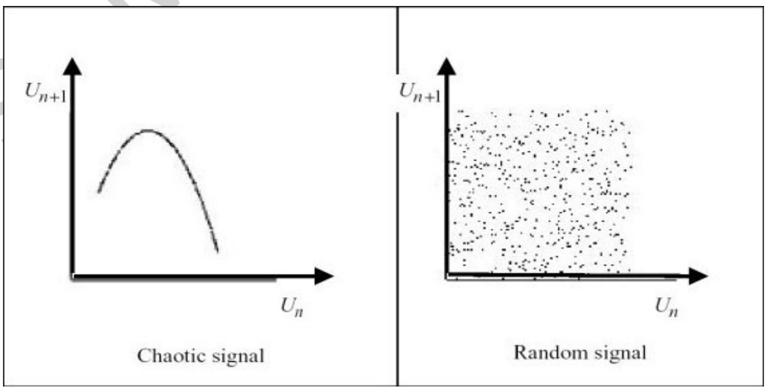

Figure 2: Chaotic and random signals

This illustrates the essential difference between chaotic signals and random signals. The ideal mathematical concept of randomness requires that two successive occurrence of U_n must be independent, which means that the probability of appearance of any value of U_{n+1} is not affected by the knowledge of U_n. The visual consequence of this is the apparently uniform repartition on the square of the occurrences of the points of coordinates $\left(U_{n+1}, U_n\right)$. Actually definition of perfect randomness is more stringent and requires that all finite sequences are mutually independent, but we do not go further in this direction. Completely different is the chaotic signal. Since the next U_{n+1} is related to U_n by the functional relation:

$$U_{n+1} = 3.8\, U_n \left(1 - U_n\right)$$

it is not surprising at all that all the point of coordinates $\left(U_{n+1}, U_n\right)$ are contained in the graph of the mapping:

$$U \to 3.8\, U \left(1 - U\right)$$

Notice that the “random” process of our computer is not able to realize a true mathematical pure random process, but only an approximation of it by some deterministic device.

1.2. One Dimensional Discrete Chaotic Systems

Consider a mapping *f* from some bounded subset K of R^n into itself and consider the dynamical system defined by:

$$x_{n+1} = f\left(x_n\right)$$

and denote by $x_n\left(x_o\right)$ the trajectory issued from x_o (this means the sequence issued from x_o and defined by induction with the above formula.

Definition : Sensitivity to initial conditions

One says that the system possesses the property of sensitivity to initial conditions on some subset A of K if there exist some positive number M such that for every x_o in A and for every (small) positive ε one can find some initial condition x_o' and an integer *n* such that:

$$\left\| x_n\left(x_o\right) - x_n\left(x_o'\right) \right\| \geq M$$

We illustrate this by the following experiments with the system (1) for the value 0.8 of the parameter. We have plotted the difference $\left|U_n(U_0)-U_n(U_0')\right|$ for initial conditions differing successively from 10^{-6}, 10^{-9}, 10^{-12} in Figure 3.

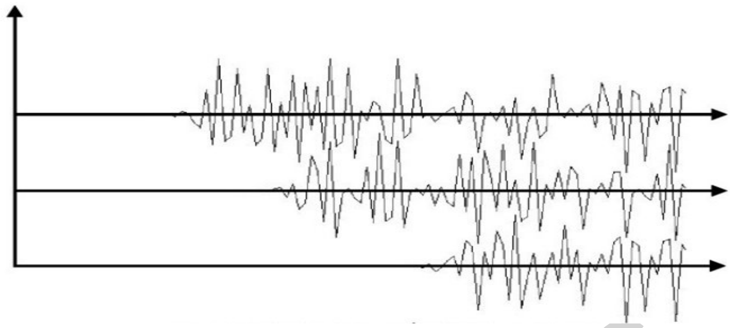

Figure 3: The function $\left|U_n(U_0)-U_n(U_0')\right|$ for different initial conditions

We see that for few iterates (15,25,35) the two trajectories are impossible to distinguish (up to the precision of the pixel of the screen) and then, abruptly, the difference becomes visible ant unpredictable. The difference grows exponentially which means that to add some constant number to the number of iterates where the trajectories are undistinguishable one has to divide by some constant factor the distance of the two initial conditions; in our example one have to divide by 1000 to obtain ten more iterations.

Definition : We say that system (2) is chaotic if it has a trajectory which is dense on a subset A which is an attractor, if it has the property of sensitivity to initial condition on subset A and, moreover, periodic orbits are dense in A.

Because of the sensibility to initial condition, what is the meaning of our numerical computations is not clear. All what we know is that our computer simulation has nothing to do with the actual trajectory ! Fortunately there is a deep mathematical result which state (in a precise mathematical way) that every computer simulation is actually close to the true trajectory issued from an initial condition which is not the one we used in the computation but is close to it. By the way, in a chaotic system, it is impossible to predict the trajectory issued from an initial condition but what we compute is a typical possible outcome of the system.

1. 3. Two Dimensional Discrete Chaotic System

So far we have considered a one dimensional chaotic system. An example of a famous chaotic system in two dimensions is the Hénon mapping, proposed in 1976. Consider the mapping:

$$(x,y)\rightarrow\left(y+1-ax^2,bx\right)$$

and the iterations:

$$x_{n+1} = y_n + 1 - ax_n$$
$$y_{n+1} = bx_n$$

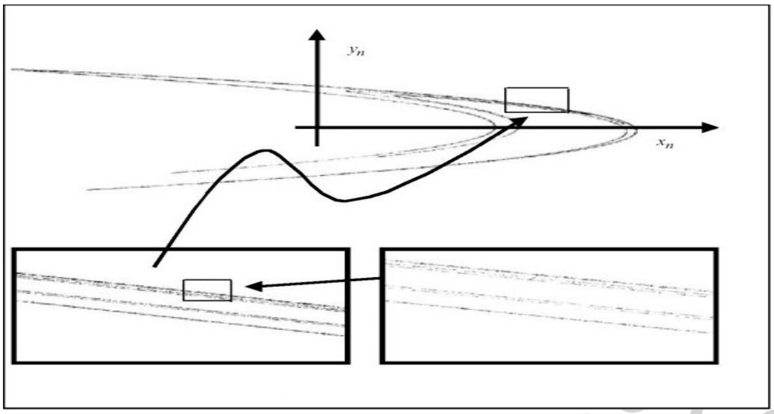

Figure 4: Two successive enlargements for the iterates of the Hénon mapping for the values $a = 1,4$ and $b = 0.3$. Figure 4 shows 10^7 iterates of this mapping starting from some initial condition.

Each enlargement is a copy of the previous picture. This is another feature of chaotic systems. The *fractal* nature of each trajectory.

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Bibliography

Allouche J-P, and C. Reder (1984), Oscillations spatio-temporelles engendrées par un automate cellulaire. Discrete Applied Mathematics 8 : 215-254.

Cartwright M.L. and J.E. Littlewood. (1945) On nonlinear equations of the second order; the equation $\ddot{y} - k\left(1 - y^2\right)\dot{y} + y = b\lambda k \cos\left(\lambda t + \alpha\right)$, k large. J; London Math. Soc. 20: 180-189

Chenciner A. (1989). Systèmes Dynamiques Différentiables. Encycopedia Univeralis.

Ementrout G-B and L. Edelstein-Keshet (1993), Cellular Automata Approach to Biological Modeling. Journal of Theoretical Biology 160 : 97-133.

Feigenbaum M. (1978), Quantitative universality for a class of nonlinear transformations. J. Stat. Physics. 19: 25-52.

Lobry C. an H. Elmoznino (2000). Combinatorial Properties of some Cellular Automata Related to the Mosaïc Cycle Concept. Acta Biotheoretica 48: 219-242.

Lorenz E. (1963). J. Atoms. Sci. 20: 130-141.

May R. (1976). Simple mathematical models with very complicated dynamics, Nature 261, 459-467

Poincaré H. (Euvres complètes. Gauthier- Vilars 1951

Smale S. (1976). Differential Dynamical Systems. Bull. A.M.S.

Wiener N., and A. Rosenblueth (1946). The mathematical formulation of the problem of conduction of impulses in a network of connected excitable elements. Archives of the Institute of Cardiology of Mexico 16 : 202-2065.

Wolfram S. (1994) Cellular Automata and Complexity. Addison-Wesley

Biographical Sketch

Claude Lobry, was born in 1943 and got a "thèse d'état" at the university of Grenoble, in optimal control, in 1972. He was then appointed as professor at the University of Bordeaux and joined the University of Nice in 1981. His main interest is mathematics applied to automatic control and natural systems where he became a specialist of controllability. During the eighties he used the tools of Nonstandard Analysis in some problems of singular perturbations of differential systems.From 1990 to now he has been very active in creating interdisciplinary teams or networks with the Centre National de la Recherche Scientifique, the Institut National de Recherche Agronomique, the Institut National de Recherche en Informatique et Automatique. Besides his scientific activities he has a strong involvement in the promotion of mathematical research in developing countries, especially Africa.