# MATHEMATICAL MODELS IN ELECTRIC POWER SYSTEMS

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## Contents



## Summary

Electric power systems are typically large complex systems spread over vast geographic areas and comprising a wide array of devices. Mathematical modeling and simulations play a major role in their design and operation. This article provides a broad overview of the physical characteristics and mathematical modeling of power systems. First, the basic electrical quantities used in the mathematical description of power systems are identified. A description of the physical structure of a typical modern power system, including the functions of its major components, is then presented. The performance

requirements of a properly designed power system and the various levels of controls used to meet some of the requirements are also discussed. This is followed by a description of the physical characteristics and mathematical models of individual components. Finally, the modeling of the integrated power system and the simulation of the different aspects of its performance normally carried out are discussed. The need for making judicious simplifying approximations and analyzing specific aspects of system performance using the appropriate degree of detail of modeling is highlighted.

## 1. Introduction

Electric power systems are an integral part of the way of life in modern society. The electricity supplied by these systems has proved to be a very convenient, clean and safe form of energy. It runs our factories, warms and lights our homes, cooks our food, and powers our computers.

Electricity is carrier of energy. Energy is neither naturally available in the electrical form nor is it consumed directly in that form. The advantage of the electrical form of energy is that it can be transported and controlled with relative ease and with a high degree of efficiency and reliability. An electric power system generally refers to the collection of components interconnected to undertake the entire process of converting various primary sources of energy (hydro, fossil, nuclear, etc.) to electrical energy, transmitting it to points of consumption, and driving various power utilization devices.

The electric power industry began in the 1880s and has evolved into one of the largest industries. Large interconnected power systems have been formed in many parts of the world, covering vast geographical areas. These systems provide power to millions of industrial, commercial, and residential users with very high quality and reliability and at great affordability. To achieve this, power systems are designed and operated with well established criteria and procedures based on a wide range of engineering analyses, which require mathematical models appropriate for meeting the objectives of specific studies.

Electric power systems are predominantly three-phase AC (alternating current) systems. As opposed to DC (direct current) systems, AC systems are more convenient for generation, transmission and consumption. In an AC system, voltage levels can be easily transformed, thus providing the flexibility of using different voltages for transmission, generation, and utilization; from the viewpoints of efficiency and power-transfer capability, the transmission voltages have to be high, but it is not practically feasible to generator and consume power at these voltages. As well, AC machines (generators and motors) are simpler and cheaper than DC machines. In a power system, electrical power is generated and transmitted in a balanced three-phase system. Industrial loads are invariably three-phase; single-phase residential and commercial loads are distributed nearly equally among the phases so as to effectively form a balanced three-phase system.

The following sections describe the physical features of power systems, and introduce the mathematical models and simulations commonly used in engineering analyses of the steady state and dynamic performance of power systems.

## 2. Basic Concepts

Before the physical characteristics and modeling of power systems are discussed in detail, various electrical quantities associated with AC networks and their mathematical representation will be outlined in this section. In addition, the concepts of active power, reactive power and complex power are introduced. A clear conceptual understanding of these quantities is essential in the development and application of mathematical models of power systems.

### 2.1. Basic Electrical Quantities

In the normal steady-state operation, the wave forms of voltages and currents in an AC network are ideally sinusoidal functions of time. A sinusoidal voltage function may be written as

$$v(t) = V_{\max} \cos \omega t \tag{1}$$

The amplitude or peak value of the sinusoid is $V_{\max}$ (in volts) and the angular frequency is $\omega$ (in radians per second). The function repeats itself every $T$ seconds, with

$$T = 2\pi/\omega \tag{2}$$

In one second the function goes through $1/T$ cycles or periods. The frequency in cycles per second, or hertz (abbreviated Hz) is then

$$f = \omega/2\pi \tag{3}$$

In a power system, all voltages and currents in the AC network at a steady state have the same frequency, but are generally displaced from each other in time phase. Therefore, a more general expression for the voltage is

$$\begin{aligned} v(t) &= V_{\max} \cos(\omega t + \theta) \\ &= \sqrt{2}\quad V \cos(\omega t + \theta) \end{aligned} \tag{4}$$

where $\theta$ is the phase angle or simply the phase of the voltage, and $V$ is the effective or root mean square (rms) value of the voltage. Since the frequency is known and common to all voltages and currents in the network, the voltage $v(t)$ is completely specified by its amplitude $V_{\max}$ or $V$ and its phase $\theta$. Using the *phasor representation* in the polar form, the voltage may be expressed as

$$\tilde{V} = Ve^{j\theta} = V\angle\theta \tag{5}$$

The phasor representation in the rectangular (Cartesian) form is

$$\tilde{V} = V\cos\theta + jV\sin\theta \tag{6}$$

The real quantities are time-domain functions, and their phasors are frequency-domain functions. To solve time-domain problems in the steady state, phasors can be used and the corresponding frequency domain problems solved; this makes analysis much easier.

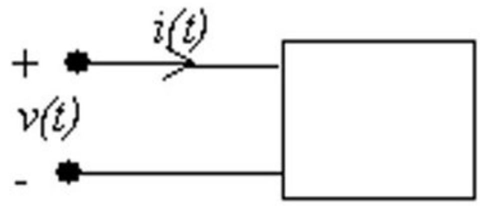


Figure: 1 General Electrical Circuit

Let us now consider a general circuit with two accessible terminals as shown in Figure 1.

If the instantaneous values of the voltage and current at the terminals are given by

$$\begin{aligned} v(t) &= \sqrt{2}\,V\cos(\omega t+\theta) \\ i(t) &= \sqrt{2}\,I\cos(\omega t+\alpha) \end{aligned} \tag{7}$$

then the phasor quantities at the terminals are

$$\begin{aligned} \tilde{V} &= V\angle\theta \\ \tilde{I} &= I\angle\alpha \end{aligned} \tag{8}$$

We define the ratio of the phasor voltage to the phasor current as the *impedance* of the circuit which we denote by *Z*. That is,

$$Z = \frac{\tilde{V}}{\tilde{I}} = \frac{V}{I}\angle(\theta-\alpha) \tag{9}$$

The above equation looks very much like the Ohm's law. Impedance, being the ratio of voltage to current, is a complex number and is measured in ohms. In the rectangular form, it is denoted by

$$Z = R + jX \tag{10}$$

where *R* is the resistive component or simply the *resistance*, and *X* is the reactive component or *reactance*.

The reciprocal of impedance, denoted

$$Y = \frac{1}{Z} = G + jB \qquad (11)$$

is called *admittance* and is measured in mhos. Impedance and admittance are complex numbers, but are not phasors; that is, they have no sinusoidal time-domain functions of any physical meaning.

The impedance of a resistor is purely resistive, its reactance being zero. Impedances of ideal inductors and capacitors are purely reactive, having zero resistive components. The impedance of an element with an inductance *L* is

$$Z_L = jX_L = j\omega L \qquad (12)$$

and the impedance of an element with a capacitance *C* is

$$Z_C = -jX_C = -j\,\frac{1}{\omega C} \qquad (13)$$

**2.2. Power in an AC Circuit**

Referring to Figure 1, the instantaneous power into the circuit is

$$p(t) = v(t)\ i(t)$$

$$= V_{\max}\, I_{\max} \cos(\omega t + \theta)\cos(\omega t + \alpha)$$

$$= \frac{1}{2} V_{\max} I_{\max} \left[\cos\phi - \cos(2\omega t - \phi)\right]$$

where $\phi = \theta - \alpha$, the angle by which the voltage phasor leads the current phasor, commonly referred to as the *power factor angle*.

If the effective (rms) values of voltage and current are used instead of the amplitudes,

$$p(t) = VI\cos\phi - VI\cos(2\omega t - \phi)$$

$$= VI\cos\phi(1 - \cos 2\omega t) - VI\sin\phi\sin 2\omega t \qquad 14)$$

$$= P(1 - \cos 2\omega t) - Q\sin 2\omega t$$

where

$$P = VI \cos \phi \tag{15}$$

$$Q = VI \sin \phi \tag{16}$$

The instantaneous power $p(t)$ thus has two components:

$$p_p = P\left(1 - \cos 2\omega t\right)$$
$$p_q = Q\left(\sin 2\,\omega t\right)$$

The component $p_p$ has an average value of $P$=*VI cos* $\phi$, and represents the component of power utilized for permanent irreversible consumption, e.g. conversion to heat or light. The quantity *P* is referred to as the *active power or real power*.

In contrast, the component $p_q$ has a zero average value and represents power utilized in establishing and discharging magnetic fields in inductors and electrostatic fields capacitors twice every cycle. In a network this interchange of energy takes place between the source, inductive elements and capacitive elements. The net energy associated with the reactive power is the sum of various inductive and capacitive stored energies. Although $p_q$ has a zero average value, it does nevertheless represent real reciprocating energy that must be present by virtue of the inductive and capacitive elements of the network. The quantity $Q$=$VI$ sin $\phi$ corresponds to the peak value of $p_q$ and is referred to as the *reactive power*.

This leads us next to the concept of *complex power S*, which is defined as the product of $\tilde{V}$ and the conjugate of $\tilde{I}$. Thus

$$S = \tilde{V}\,\tilde{I}^* = V\angle\phi\left(I\angle -\alpha\right)$$

$$= VI\,\angle\phi - \alpha = VI\,\angle\phi = VI\cos\phi + j\,VI\,\sin\phi$$

$$= P + jQ \tag{17}$$

The reason for using the conjugate of *I* in the above is to arrive at the agreed upon convention for reactive power *Q*, which is considered positive with *I* lagging *V*. By convention, *Q* absorbed by an inductive load is considered positive and by a capacitive load negative. The magnitude of the complex power *S* is used to specify the MVA (mega voltamperes) rating of equipment.

*P* and *Q* are scalar quantities. The unit of *P* is the watt. Although dimensionally, *P* and *Q* have the same units, the unit of *Q* used in practice is defined as the var (voltampere reactive) to distinguish it as the unit used for measuring reactive power.

The principle of conservation of energy requires that a balance exist between generated and consumed active (or real) power. By analogy a similar statement can be made about the reactive power. At any junction or node in a network, the total injected *P* and

*Q* are equal to the total *P* and *Q* extracted. Thus, *P* and *Q* are very useful as working quantities as they can be added arithmetically, as opposed to phasor addition required for currents and voltage. For the power system as a whole, a "balance sheet" of active and reactive power can be drawn.

At first glance the concept of reactive power may appear mysterious and obscure. In fact, it plays a very significant role in the modeling and analysis of power systems.

The flows of active power and reactive power in a transmission network are fairly independent of each other and are influenced by different control actions. The active power transfer depends mainly on the angle by which the sending end voltage leads the receiving voltage. Reactive power transfer depends mainly on the difference in voltage magnitudes. Active power control is closely related to frequency control, whereas reactive power control is closely related to voltage control. Efficient and reliable operation of power systems requires considerations to both active and reactive power controls.

## 3. Elements of an Electric Power System

Figure 2 illustrates the basic elements of a modern power system. Electric power is produced at generating stations (GS) and transmitted to consumers through a complex network of individual components, including transmission lines, transformers and switching devices

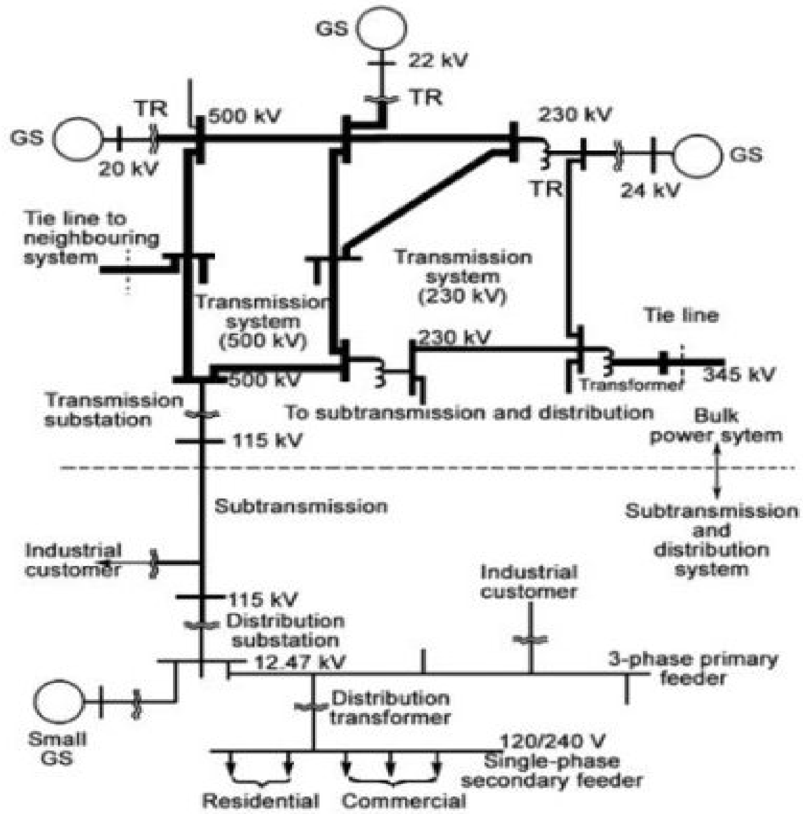


Figure 2: Basic Elements of a Power System

The overall power system consists of multiple generating sources and several layers of transmission networks. This provides a high degree of structural redundancy that enables the system to withstand unusual contingencies and outages without service disruption to the consumers.

The following is a description of the functions and physical characteristics of individual components.

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**Biographical Sketches**

**Dr. Kundur,** holds a Ph.D. in Electrical Engineering from the University of Toronto and has over 30 years of experience in the electric power industry. He is currently the President and CEO of Powertech Labs Inc., the research and technology subsidiary of BC Hydro. Prior to joining Powertech in 1993, he worked at Ontario Hydro for 25 years and was involved in the planning, design and operation of power systems.He has served as Adjunct Professor at the University of Toronto since 1979 and at the University of British Columbia since 1994. He is the author of the book Power System Stability and Control (McGraw-Hill, 1994), which is the standard modern reference for the subject. He has performed extensive international consulting and has delivered technical courses for utilities and universities around the world.Dr. Kundur is a Fellow of the IEEE, and is currently Chair of the Power System Dynamic Performance Committee of the Power Engineering Society. He is also very active in CIGRE and is currently the Chair of the Study Committee C-4 on “System Technical Performance”. He is the recipient of the 1997 IEEE Nikola Tesla Award and the 1999 CIGRE Technical Committee Award. In 2003, he was inducted as a Fellow of the Canadian Academy of Engineers.

**Dr. Wang,** holds a Ph.D. in Electrical Engineering from the University of Toronto and has over 15 years of experience in the electric power industry. He worked for Ontario Hydro before joining Powertech Labs Inc. in 1993 where he is now a principal engineer.Dr. Wang has been involved with the development of mathematical models and computer software used in the electric power industry. He is the recipient of the 1994 IEEE PES Walter Fee Outstanding Young Engineer Award.