MATHEMATICAL MODELING IN SOCIAL AND BEHAVIORAL SCIENCES

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Summary

This chapter presents some applications of mathematics in social and behavioral sciences. It provides a general overview of mathematical approaches to different social and behavioral problems. Since the literature on the topic is vast, we can illustrate only a few areas of applications in this chapter. We introduce applications of a few branches of mathematics – including optimization theory, operations research, game theory, differential equations, and chaos theory. To depict how these theories can be applied to social and behavioral sciences, in each case we solve one simple example of a problem of social/behavioral sciences by applying one of these branches of mathematics. Firstly, we apply optimization theory and comparative statics analysis to decide the optimal behavior of workers with multiple interests and examine the impact of job amenity on moonlighting. Secondly, operations research is applied to solve the job assignment problem. Thirdly, we examine political competition, applying game theory. Then, as an example of applying differential equations, we investigate long-run socioeconomic consequences of altruism and explain under what conditions the opposite opinions of Malthus and Keynes on the issue are valid. Finally, chaos theory is applied to identify socioeconomic chaos from a simple one-dimensional population model built on Malthus' population theory.

1. Introduction

Mathematics is an integral part of contemporary social and behavioral sciences. Many of today's profound insights into human behavior could hardly be obtained without the help of mathematics. It may be said that the main advance in modern social and behavioral sciences is characterized by applying mathematics to various social and behavioral problems. The concepts of equilibrium versus non-equilibrium, stability versus instability, and steady states versus chaos in the contemporary literature are difficult to explain without mathematics. In some sense, one can hardly properly appreciate achievements of contemporary social and behavioral sciences without a thorough training in mathematics.

The purpose of this chapter is to present some mathematical models that support the field of social and behavioral sciences. It provides a general overview of mathematical approaches to different social and behavioral problems. The variety of models as well as the number of recent contributions is quite impressive. Since the literature on the topic is vast, we will provide only a few areas of the applications in this chapter. We introduce examples of applications of a few branches of mathematics – including optimization theory, operations research, game theory, differential equations theory, and chaos theory. Section 2 applies optimization theory and comparative statics analysis to decide the optimal behavior of workers with multiple interests and examine the impact of job amenity on moonlighting. Section 3 applies operations research to solve the job assignment problem. Section 4 examines political competition, applying game theory. As an example of applying differential equations, Section 5 investigates long-run socioeconomic consequences of altruism and explains under what conditions the opposite opinions of Malthus and Keynes on the issue are valid. Section 6 shows how chaos theory can be applied to identify socioeconomic chaos from a simple one-dimensional population model built on Malthus' population theory. The bibliography at the end of the chapter provides the reader with guidelines for further investigations.

2. Optimization Theory – Job Amenity and Moonlighting

Optimization is a branch of mathematics, which encompasses many diverse areas of minimization and maximization. It enables us to determine the most profitable or least disadvantageous choice out of a set of alternatives. Typically the set of alternatives is restricted by several constraints on the values of a number of variables and an objective function locates an optimum in the remaining set. The method is largely used in operations research and systems analysis, that is for example, for optimal scheduling of production processes, for determining the best way for transporting a certain commodity. Optimization theory is closely related to the calculus of variations, control theory, convex optimization theory, decision theory, game theory, linear programming, Markov decision chains, network analysis, optimization theory, queuing systems, etc.

This section uses an example of how people rationally determine their occupation(s) to illustrate a generic process of applying optimization theory to solve a decision-making problem. Different jobs, such as officer, police, university professor, factory worker, are associated with different social status and amenities. Differences in amenities give rise to compensating wage differentials. Assuming that different jobs bring about different amenities and disamenities, Lundborg (1995) builds some models to explain why some people would have multiple occupations. We now introduce Lundborg's general model of amenities and moonlighting to show how traditional optimization theory can be applied to explain behavior of workers.

Assume two sectors, one with jobs with some amenity and one traditional manufacturing sector with no amenity, denoted as the a-sector and the m-sector, respectively. We express a typical individual's utility function as $U = U(X, a, l)$, where X is his commodity consumption, a is the share of his total available time spent for work in the amenity sector and l is the share of time for leisure.

Assume that the total time T is divided between leisure time L, work time in the a-sector, A, and work time in manufacturing, M so that

$$T = L + A + M .$$

We thus have

$$l + a + m = 1 \tag{2.1}$$

where $l \equiv L/T$, $a \equiv A/T$, and $m \equiv M/T$. Assume that U rises in all three arguments. Consumption X depends on the individual's earnings such that

$$X = w_{\mathrm{m}}(1 - a - l) + wSa \tag{2.2}$$

where we use (2.1), w_{m} is the individual's wage in manufacturing work, w is his wage in the a-sector and $S \equiv (1 + s)$ where s is a subsidy rate provided by the government. We assume that w_{m} and w are fixed for the individual and $w_{\mathrm{m}} > Sw$. Since the job with amenity provides more pleasure than the job with no amenity, it is reasonable to require that the wage rate in the sector with no amenity is higher than the "net wage rate", Sw, in the sector with amenity. Otherwise, all the time available for work would be spent in the sector with amenity. The individual's rational behavior is described by the following maximization problem

$$\underset{X, a, l}{\text{Maximize}}\ U = U(X, a, l) \tag{2.3a}$$

subject to

$$w_{\mathrm{m}} = X + (w_{\mathrm{m}} - wS)a + w_{\mathrm{m}} l , \tag{2.3b}$$

$$0 \le a,\ l \le 1. \tag{2.3c}$$

There are then three goods, X, a and l, with prices 1, $(w_{\mathrm{m}} - Sw)$, and w_{m} respectively. Define the Lagrange function Γ

$$\Gamma(X, a, l, \lambda) \equiv U(X, a, l) + \lambda\left[w_{\mathrm{m}} - X - (w_{\mathrm{m}} - wS)a - w_{\mathrm{m}} l\right]$$

where λ is a Lagrangian multiplier. The first-order conditions for the maximization problem are obtained as

$$\frac{\partial \Gamma}{\partial X} = \frac{\partial U}{\partial X} - \lambda = 0, \tag{2.4}$$

$$\frac{\partial \Gamma}{\partial a} = \frac{\partial U}{\partial a} - \lambda(w_{\mathrm{m}} - wS) = 0, \tag{2.5}$$

$$\frac{\partial \Gamma}{\partial l} = \frac{\partial U}{\partial l} - \lambda w_{\mathrm{m}} = 0, \tag{2.6}$$

$$\frac{\partial \Gamma}{\partial \lambda} = w_{\mathrm{m}} - X - (w_{\mathrm{m}} - wS)a - w_{\mathrm{m}} l = 0. \tag{2.7}$$

Calculating the total differentials of Eqs. (2.4)-(2.7) with w_{m}, w and S as parameters and expressing the results in matrix form yields [This form is commonly used in social sciences. See Chiang (1984)],

$$D\begin{bmatrix} dX \\ da \\ dl \\ d\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda dw_{\mathrm{m}} - \lambda S dw - \lambda S dS \\ \lambda dw_{\mathrm{m}} \\ \lambda_1 dw_{\mathrm{m}} - Sadw - wadS \end{bmatrix} \tag{2.8}$$

where $a_1 \equiv l + a - 1$ and

$$D \equiv \begin{bmatrix} U_{XX} & U_{Xa} & U_{Xl} & -1 \\ U_{Xa} & U_{aa} & U_{al} & -w_0 \\ U_{lX} & U_{la} & U_{ll} & -w_{\mathrm{m}} \\ -1 & -w_0 & -w_{\mathrm{m}} & 0 \end{bmatrix}$$

where the variables with double suffixes are second (partial) derivatives with the variable(s) and $w_0 \equiv w_{\mathrm{m}} - wS$. It is known that a sufficient condition for a solution of the first-order conditions to be a maximum is that the determinant $|D|$ of the matrix is negative[See Chiang (1984)]. This is guaranteed if the utility function is strongly concave in X, a, and l. We require this condition to be satisfied in the remainder of this section.

We first examine the effects on the share of the work time spent in the amenity sector as a result of an increase in the manufacturing wage. With $dw = dS = 0$, we rewrite (2.8) as

$$D \begin{bmatrix} \dfrac{dX}{dw_{\mathrm{m}}} \\ \dfrac{da}{dw_{\mathrm{m}}} \\ \dfrac{dl}{dw_{\mathrm{m}}} \\ \dfrac{d\lambda}{dw_{\mathrm{m}}} \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \\ \lambda \\ a_1 \end{bmatrix}$$

This is a linear equation. Applying Cramer's rule, we solve for the impact of a change w_{m} on a as follows

$$\frac{da}{dw_{\mathrm{m}}} = \frac{U_{xx}W - \lambda\left(U_{XX} w_{\mathrm{m}} wS + U_{ll} - U_{al}\right)}{\det D} \tag{2.9}$$

where

$$W \equiv \left[w_0\left(a_1 U_{ll} + \lambda w_{\mathrm{m}}\right) - a_1 w_{\mathrm{m}} U_{al}\right] U_{XX} .$$

The effect of an increase in manufacturing wages on part-time in the a-sector is ambiguous. In line with the *Slutsky equation*, the effect can be broken down into an income effect and a substitution effect. An increase in manufacturing wages implies that the income level has increased. As the amenity sector work is a normal good, there is a tendency to spend a larger share of total work in the a-sector. On the other hand, as the manufacturing wage w_{m} rises, consumption of amenity becomes more costly since the price of amenity sector work, $w_{\mathrm{m}} - wS$, rises. Since the two effects are opposite, the net effect is ambiguous.

Similarly, the effects of an increase in w_{m} on consumption of X are given by,

$$\frac{dX}{dw_{\mathrm{m}}} = \frac{\lambda_0 + U_{aa}\left(a_1 U_{ll} + \lambda w_{\mathrm{m}}\right) - a_1 U_{al}^2 - \lambda w_0 U_{al}}{\det D} > 0, \tag{2.10}$$

in which

$$\lambda_0 \equiv \lambda\left(w_0 U_{ll} - U_{la} w_{\mathrm{m}}\right).$$

As the manufacturing wage is increased, consumption of the commodity is increased. There is a positive income effect on X as w_{m} rises. As consumption of amenity work becomes more costly as the price of a-sector work rises, the individual substitute consumption of the a-sector work with consumption of X .

The effects on leisure of increases in w_{m} are

$$\frac{dl}{dw_m} = \frac{U_{XX}\left[a_1 w_m U_{aa} - w_0^2\left(1 + w_m\right)\lambda - a_1 w_0 U_{al}\right] + \left(U_{al} - U_{aa}\right)\lambda}{\det D}. \tag{2.11}$$

The effects are ambiguous. The income effect is positive, but the substitution one is negative.

We can similarly carry out the comparative statics analysis with regard to the a-sector wage and the subsidy rate. This example shows the ‘standard procedure’ for analyzing rational behavior by optimization theory. First we describe mathematically rational behavior of individuals as an optimization problem. Then, we find the first-order conditions and solve the associated equations. Finally we check the second-order conditions and examine the impact of changes in some parameters.

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Biographical Sketch

Dr. Wei-Bin Zhang, born in 1961, teaches and does research at Ritsumeikan Asia Pacific University in Japan. His main research fields are nonlinear socioeconomic dynamics, social science, regional and international economics, urban economics, East Asian economic development, Chinese philosophy, and ethics. He has published more than 100 academic articles. His books include Economic Dynamics (1990), Synergetic Economics (1991), Japan versus China in the Industrial Race (1998), Confucianism and Modernization (1999), and Capital and Knowledge (1999), Adam Smith and Confucius - The Theory of Moral Sentiments and The Analects (2000), A Theory of International Trade (2000), An Economic Theories of Cities (2002), The American Civilization Portrayed in Ancient Confucianism (2003), Economic Growth Theory (2005), and Discrete Dynamical Systems, Bifurcations and Chaos in Economics (2006).

He was brought up and received his university education in mainland China. He studied at Kyoto University in Japan for four years. He conducted research in Sweden from 1987 to 1998.