BASIC METHODS FOR SOLVING EQUATIONS OF MATHEMATICAL PHYSICS

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Summary

In mathematical physics models of various physical processes are studied. The basic mathematical objects are partial differential equations, integro-differential and integral equations. As a rule, a mathematical model has the form of a boundary value or initial boundary value problem for an equation or a system of equations. Elliptic, hyperbolic and parabolic equations define the major classes of these problems. We recognize classical and generalized (weak) formulations of these problems. An important concept of a generalized solution is based on the notion of a generalized derivative with the use of the Sobolev spaces. The theory of differential and integral equations, the theory of functions and embedding theorems, methods of functional analysis, approximate methods and computational mathematics are the main tools for studying problems of mathematical physics. In this chapter basic methods for solving problems of mathematical physics are presented: potential methods, eigenvalue problems, methods of integral transforms, methods of transformation groups (analytical methods); discretization methods, methods for nonlinear problems, variational methods (approximate methods).

Introduction

Active development of productive forces in the countries of Western Europe late in the 17th century and early in the 18th century became the main reason for significant progress in natural sciences and engineering, including celestial mechanics, hydrodynamics, dynamics of a system of material points and the theory of electricity, magnetism, and heat. On the basis of these sciences there arose directions of research and problems which subsequently led to basic problems, notions and theorems of mathematical physics. For example, two problems closely related to each other arose in celestial mechanics: the problem on the shape of the Earth and other planets and the problem on their mutual attraction. The former problem led to the complicated theory of the shape of a planet (this theory is important in navigation, geography and geodesy) and the latter provided the basis for the development of the theory of attraction which became the foundation for the theory of Newtonian potential. In 1686 Newton stated its basic principles in his famous “Mathematical Elements”. Further investigations concerned with attraction of spheroids are due to Laplace and Lagrange. In 1765 Euler introduced the notion of a potential for problems of hydrodynamics. The concept of a potential as a function whose gradient is a vector field is due to Gauss. Coulomb and Poisson were the first to study properties of a simple layer potential. An important contribution to the development of the potential theory was made by Green starting in 1828 when he began to study boundary value problems. However, the ideas of the potential method appeared to be very useful in the theory of elliptic and parabolic equations and of systems of a general form, in particular, when studying existence and smoothness of solutions of linear and nonlinear boundary value problems. These investigations were conducted by scientists of different countries in the second part of
the 20th century and are still in progress.

A new stage of the development of mathematical physics began in the 20th century when problems of the theory of relativity, quantum mechanics, hypersonic gas dynamics, kinetic equations, the theory of nuclear reactors, plasma physics, biology, and economics were involved in this field of science. Many problems of mathematical physics are reduced to boundary value problems for differential (or integro-differential) equations which, combined with boundary (and initial) conditions, form mathematical models of physical processes. The theory of partial differential equations, the theory of integral equations, the theory of functions and functional analysis, approximate methods are basic mathematical tools for studying these problems.

By convention methods for solving problems of mathematical physics can be divided into two wide classes. Analytical methods that include construction and justification of exact solutions in the form of finite formulae or series, problems of existence, uniqueness, and smoothness of a solution, form the first class. The second class involves approximate methods: discretization methods, splitting methods, iterative methods for solving nonlinear problems.

1. Analytical Methods for Problems of Mathematical Physics

The term analytical methods implies the study and rigorous derivation of finite formulae and analysis of the inner mathematical structure of a model. In both cases there arise questions which lead to more general mathematical problems: existence and uniqueness of a solution, integral representation of a solution, representation of a solution as a series etc. These problems are not immediately related to some special model.

1.1. Methods of the Potential Theory

1.1.1. Green’s Formulae

Green’s formulae are used for calculation of an integral of a function of many variables and relate the value of an \( n \)-multiple integral over a domain \( \Omega \in R^n \) to that of an \((n-1)\)-multiple integral over a piecewise smooth boundary \( \Gamma \) of this domain. The simplest formula has the form

\[
\int_{\Omega} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\Gamma} (P dx + Q dy), \quad \Omega \subset R^2. \tag{1}
\]

A domain \( \Omega \) is assumed to be oriented in a natural way. Formula (1) (known to Euler in the 18th century) has a simple hydrodynamical sense: the flow of a liquid, moving over a plane with velocity \( \mathbf{v} = (Q, -P) \) across a boundary \( \Gamma \), is equal to the integral over a domain \( \Omega \) of intensity (divergence) \( \text{div} \mathbf{v} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \) of sources and sinks distributed in \( \Omega \).

The following formulae are due to Green (1828):
\begin{equation}
\int_{\Omega} \left( v \Delta u + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \frac{\partial v}{\partial y} \right) dx = \int_{\Gamma} \nu \frac{\partial u}{\partial N} d\Gamma \tag{2}
\end{equation}

and

\begin{equation}
\int_{\Omega} \left( v \Delta u - v \Delta u \right) dx = \int_{\Gamma} \left( \frac{\partial v}{\partial N} - \frac{\partial u}{\partial N} \right) d\Gamma \tag{3}
\end{equation}

Here \( \Omega \subset R^3 \), \( x=(x_1,x_2,x_3) \), \( dx=dx_1 \, dx_2 \, dx_3 \), \( N=(N_1,N_2,N_3) \) is a unit outer normal to \( \Gamma \), \( \partial/\partial N = \sum_{i=1}^{3} N_i \partial/\partial x_i \), \( \Delta = \sum_{i=1}^{3} \left( \partial/\partial x_i \right)^2 \) is the Laplace operator. Formulae (2) and (3) are also valid for \( \Omega \subset R^n \).

The generalization of formulae (2) and (3) for linear partial differential operators, provided that their coefficients are sufficiently smooth, has the following form:

1) if

\begin{equation}
Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u,
\end{equation}

are adjoint real operators of the second order, \( a_{ij} = a_{ji} \), then

\begin{equation}
\int_{\Omega} \left[ vLu + \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_i} \frac{\partial u}{\partial x_j} \right] dx = \int_{\Gamma} \nu \frac{\partial u}{\partial M} d\Gamma,
\end{equation}

\begin{equation}
\int_{\Omega} \left( Lu - vLu \right) dx = \int_{\Gamma} \left( \frac{\partial v}{\partial M} - \nu \frac{\partial u}{\partial M} - Bu \right) d\Gamma,
\end{equation}

where \( B = \sum_{i=1}^{n} b_i N_i \), \( \partial/\partial M = \sum_{i,j=1}^{n} a_{ij} \partial/\partial x_i \) is a derivative with respect to a conormal,

\begin{equation}
M = \left\{ \sum_{j=1}^{n} a_{1j} \partial/\partial x_j, ..., \sum_{j=1}^{n} a_{nj} \partial/\partial x_j \right\};
\end{equation}

2) if

\begin{equation}
Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u,
\end{equation}
\[ L^* v = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x)v) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (b_i(x)v) + c(x)v, \]

then

\[
\int_{\Omega} \left[ vLu + \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} - \left( \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu \right) v \right] \, dx = \int_{\Gamma} v \frac{\partial u}{\partial M} \, d\Gamma,
\]

\[
\int_{\Omega} (uL^* v - vLu) \, dx = \int_{\Gamma} \left( v \frac{\partial u}{\partial M} - v \frac{\partial u}{\partial M} - Cuv \right) \, d\Gamma,
\]

where \( M \) is a conormal and \( C = \sum_{i=1}^{n} N_i \left( b_i - \sum_{j=1}^{n} \partial a_{ij} / \partial x_j \right) \).

3) if

\[ Lu = \sum_{p=1}^{m} \sum_{|\alpha|=p} a_{\alpha}(x) D^\alpha u + a(x)u, \quad L^* v = \sum_{p=1}^{m} (-1)^p \sum_{|\alpha|=p} D^\alpha (a_{\alpha}(x)v) + a(x)v \]

are adjoint real operators of order \( m, \alpha=(\alpha_1, \alpha_2, \ldots, \alpha_p) \) is an integer multi-index of length \( |\alpha|=p \), \( 1 \leq \alpha_i \leq n \), \( D^\alpha = \partial^{\alpha_1}_{\alpha_1} \partial^{\alpha_2}_{\alpha_2} \cdots \partial^{\alpha_p}_{\alpha_p} \), \( \partial_i = \partial / \partial x_i \), then

\[
\int_{\Omega} \sum_{p=1}^{m} \sum_{|\alpha|=p} \sum_{k=1}^{p} \int_{\Gamma} (-1)^k \left[ \partial^{\alpha_k} \cdots \partial^{\alpha_{k-1}} (a_{\alpha})v \right] x \times N_{\alpha_k} \left[ \partial_{\alpha_{k+1}} \cdots \partial_{\alpha_p} u \right] \, d\Gamma. \quad (4)
\]

Here the integral over a boundary can be written as a bilinear form \( \sum_{i,j} B_{ij} (S_i \, u)(T_j \, v) \, d\Gamma \) with some differential operators \( S_i, T_j \) of order \( s_i, t_j \), \( 0 \leq s_i + t_j \leq m - 1 \), respectively.

Green’s formulae play an important role in calculus and, especially, in the theory of boundary value problems for differential and partial differential operators of the second (and higher) order.

For functions \( u(x), v(x) \) being sufficiently smooth in \( \Omega \cup \Gamma \), where \( \overline{\Omega} = \Omega \cup \Gamma \), these formulae lead to some relations useful in studying properties of a solution of a boundary value problem, in determining the kind of a boundary value problem, in deriving a solution in an explicit form, and in establishing \( a \text{ priori} \) estimates of a solution up to a boundary.
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Biographical Sketch

V.K. Andreev was born in village Kalinino of Nerchinsk region, Russia. He completed his Diploma in Mechanics and Applied mathematics at the Novosibirsk State University in 1972. In 1975 he took the Russian degree of Candidate in Physics and Mathematics at the Institute of Mathematics of the USSR Academy of Sciences, Novosibirsk. In 1990 V.K. Andreev was awarded the Russian degree of Doctor in Physics and Mathematics from Institute of Hydrodynamics of the USSR Academy of Sciences, Novosibirsk with the thesis “Invariant solutions of the hydrodynamics equations with a free boundary and their stability”. In 1992 he took Professor Diploma at the Chair “Mathematical analysis and differential equations”. From 1975 to 1990 he was a senior worker at the Computer Center of the USSR Academy of Sciences in Krasnoyarsk, Russia (now renamed the Institute of Computational Modeling of the Russian Academy of Sciences). Since 1990 V.K. Andreev is Head of Department “Differential equations of mechanics” at the same institute. V.K. Andreev is a well-known specialist in the fields of computational hydrodynamics, group analysis and hydrodynamic stability. He authored five monographs and over 170 scientific papers, devoted mathematical modeling, numerical methods of solving boundary value problems, group classification and hydrodynamic stability.