NUMERICAL METHODS FOR INTEGRAL EQUATIONS

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Summary

This chapter presents a review of direct and iterative numerical methods for solving linear and nonlinear integral equations of the second kind.

1. Introduction

An integral equation is an equation with an unknown function under the integral sign. In a general case an integral equation is of the form

$$\int_a^b K\left(x,s,u\left(s\right)\right)ds = f\left(x,u\left(x\right)\right), \quad a \le x \le b. \tag{1.1}$$

Here x is an independent variable, $u(x)$ is an unknown function, $K(x,s,u)$ is a kernel of the integral equation, $f(x,u)$ is a right-hand side, s is a variable of integration.

Many problems are reduced to integral equations (in mechanics, radio engineering, hydrodynamics, aerodynamics, electrodynamics, quantum mechanics, etc.). The integral formulation of motion equations in the form of conservation laws is also used when we construct conservative difference schemes for some problems (in particular, in mechanics of continuous media).

When solving some problems, integral equations are better to handle than differential equations. For example, the Cauchy problem

$$\frac{du}{dx} = f(x,u), \quad u(x_0) = u_0 \tag{1.2}$$

can be formulated in the form of the integral equation

$$u\left(x\right) = u_0 + \int_{x_0}^{x} f\left(s,u\left(s\right)\right)ds. \tag{1.3}$$

Thus, the setting of a problem is completely taken into account in the integral equation and there is no need to define additional conditions (initial and boundary).

Eq. (1.1) is written in the case of one independent variable x. However, it is easy to write its multi-dimensional analogue for independent variables $x = \left(x_1, x_2, ..., x_n\right)$. A multi-dimensional integral equation for some domain G in the n-dimensional space can be written as

$$\int_G K\left(x,s,u\left(s\right)\right)ds = f\left(x,u\left(x\right)\right), \quad x \in G \tag{1.4}$$

or

$$\int_G K\left(x_1,\ldots,x_n,s_1,\ldots,s_n,u(s_1,\ldots,s_n)\right)ds_1\cdots ds_n = f\left(x_1,\ldots,x_n,u(x_1,\ldots,x_n)\right),$$
$$x = \left(x_1,...,x_n\right) \in G \subset R^n.$$

Methods for solving the one-dimensional equation (1.1) are naturally extended to the case of the multi-dimensional integral equation (1.4) (one-dimensional integrals are replaced by multi-dimensional ones). By contrast, for differential equations the

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approaches and methods, which are applied in the multi-dimensional case (partial differential equations), differ radically from those in the one-dimensional case (ordinary differential equations).

From the above reasoning, further we shall restrict ourselves to the consideration of the one-dimensional equation (1.1).

Now let us show some special cases of the equation (1.1) which frequently occur in applications and are most studied.
The equation (1.1) is called a linear integral equation if an unknown function enters linearly into it. The Fredholm equation of the first kind

$$\int_a^b K(x,s)u(s)\,ds = f(x), \quad a \le x \le b \tag{1.5}$$

is an example of such equations.

The Fredholm equation of the second kind is of the form

$$u(x) - \lambda \int_a^b K(x,s)u(s)\,ds = f(x), \quad a \le x \le b. \tag{1.6}$$

In the Fredholm equations the kernel $K(x,s)$ is defined and bounded on a square $a \le x \le b,\ a \le s \le b,$ i.e., $|K(x,s)| \le C$ at all points of this square, where $C > 0$ is some constant, λ is a number parameter, $K(x,s)$ and the right-hand side $f(x)$ are given functions. If in the Fredholm equations we have $K(x,s) = 0$ for $x < s$, i.e., the kernel is nonzero only on the triangle $a \le s \le x,\ a \le x \le b,$ then the equations (1.5) and (1.6) become the Volterra equations of the first and second kinds, respectively:

$$\int_a^x K(x,s)u(s)\,ds = f(x), \quad a \le x \le b, \tag{1.7}$$

$$u(x) - \lambda \int_a^x K(x,s)u(s)\,ds = f(x), \quad a \le x \le b. \tag{1.8}$$

Further we will consider only the Fredholm equations of the second kind because the equations of the first kind are ill posed and some special methods are required for their research.

If the right-hand side of the equation (1.6) is equal to zero, then we have a homogeneous Fredholm equation of the second kind which can be written as

$$u(x) = \lambda \int_a^b K(x,s)u(s)\,ds, \quad a \le x \le b. \tag{1.9}$$

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This equation admits the zero (trivial) solution $u(x) = 0$. In this case the eigenvalue problem can be formulated. If the equation (1.9) has a nonzero solution $u = \varphi_i(x)$ for some parameter λ_i then this parameter is called an eigenvalue of the kernel $K(x,s)$ or of the equation (1.9) and the solution $\varphi_i(x)$ is called an eigenfunction.

If the kernel $K(x,s)$ is a continuous function on the square $a \le x \le b$, $a \le s \le b$ and λ is not an eigenvalue of this kernel then the non-homogeneous equation (1.6) has a unique continuous solution $u(x)$, $x \in [a,b]$, for any continuous right-hand side $f(x)$. Otherwise the non-homogeneous equation either has no solution or has an infinite number of solutions.

If the kernel $K(x,s)$ and the right-hand side $f(x)$ are continuous together with their derivatives of order p then a solution has continuous derivatives of order p.

In practical applications the Fredholm equations of the second kind with a real symmetric kernel $K(x,s)$, i.e., when $K(x,s) = K(s,x)$, play an important role.

A symmetric kernel has the following properties:
1) a symmetric kernel has at least one eigenvalue;
2) all eigenvalues of a symmetric kernel are real;
3) eigenfunctions $\varphi_i(x)$ of a symmetric kernel are orthogonal, i.e., $\int_a^b \varphi_i(x)\varphi_j\,dx = 0, \quad i \ne j.$

The Volterra equation (1.8) has no eigenvalue. The corresponding homogeneous equation with $f(x) \equiv 0$ has the trivial solution $u(x) = 0$ only. Hence, the nonhomogeneous equation (1.8) has a solution and this solution is unique for any value of λ.

So, the basic problems for the considered integral equations are the following:
1) finding a solution of a nonhomogeneous integral equation for a given value of the parameter λ;
2) the calculation of eigenvalues and finding the corresponding eigenfunctions of a homogeneous integral equation.

Both stated problems are very interesting in the theory and applications of integral equations. However, the first problem is of greater interest (in our opinion) in practical applications. Since a solution $u(x)$ of the equation (1.6) can be obtained very seldom, many different numerical methods for solving these equations are developed. In addition, the Volterra equation (1.8) can be considered as a special case of the Fredholm equation. A function $u(x)$ is called a solution of the equation (1.6) if the identity

$$u(x) - \lambda \int_a^b K(x,s)u(s)ds \equiv f(x), \quad a \le x \le b, \tag{1.10}$$

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is valid.

When considering numerical methods, we will suppose that the parameter λ is not an eigenvalue of the equation and the equation has a unique solution. Then the parameter λ can be assumed to be equal to unity that is equivalent to introducing a new kernel and we arrive at the equation

$$u(x)-\int_a^b K(x,s)u(s)ds = f(x), \quad a \le x \le b. \tag{1.11}$$

The quadrature method is widely used among numerical methods for solving integral equations.

2. Quadrature Methods

These methods are based on the use of numerical integration formulae for the calculation of definite integrals that enter into integral equations. Therefore we turn our attention to the notion of a definite integral and its calculation with the help of quadrature formulae.

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Biographical Sketches

A.M. Denisov was born in Moscow, Russia. He completed his studies with the Ph.D in Mathematics at the Lomonosov Moscow State University (LMSU) with thesis on solving integral equations of the first kind in 1972. In 1987 he obtained the Russian degree of Doctor in Physics and Mathematics at LMSU with the thesis 'Inverse problems of heat conduction, absorption, scattering and methods for solving them'. From 1972 to 1994 Dr. A. Denisov was assistant professor, associate professor and professor of the Faculty of Computational Mathematics and Cybernetics of the LMSU. Since 1994 he is Head of Department of Mathematical Physics of the Faculty of Computational Mathematics and Cybernetics of the LMSU. Prof. A. Denisov's special fields of research activity include theory of inverse problems of mathematical physics, theory of ill-posed problems and integral equations. He wrote about 100 scientific papers including 3 books. He is a member of Editorial Boards of 3 Scientific Journals.

Ivan K. Lifanov, Professor, D.Sc., Ph.D., is a noted scientist of broad interests (integral equations, aerodynamics, electromagnetic waves diffraction, elasticity theory and ecology). Ivan Lifanov graduated from the Department of Mechanics and Mathematics of the Moscow State University (MSU) in 1965. He received his Ph.D. degree in 1968 at the MSU and D.Sc. degree in 1981 at the Computing Center of the Russian (former USSR) Academy of Sciences. Since 1968 Ivan Lifanov has been a member of the staff of the Zhukovsky Air Force Engineering Academy (AFEA) (Moscow) where he is at present the Head of the Department of Mathematics. At present he is also chief researcher of the Institute of Computing Mathematics of the Russian Academy of Sciences (Moscow) and professor of the Orel State University (Orel). Since 1996 he is a Honoured Science Worker of the Russian Federation, since 1998 is a Honorary Professor of the AFEA and since 2000 is a vice-chairman of the Scientific Methodical Council on Mathematics of Education Ministry. Professor Lifanov is a member of the Editorial Board of the journals "Differential Equations " and " Electromagnetic Waves and Electronic Systems". Professor Lifanov is an author of over 250 scientific publications and 8 monographs.

Eugenyi V. Zakharov was born in Moscow, Russia. He completed his Diploma in Physics and Applied Mathematics at Department of Physics of Moscow State University in 1963. In 1967 he obtained the Russian degree of Candidate in Physics and Mathematics. In 1974 E.V. Zakharov was awarded the Russian degree of Doctor in Physics and Mathematics from Moscow State University with the thesis "Integral equations method for computational research of electromagnetic fields in the homogeneous media". In 1981 E.V.Zakharov obtained the Diploma of Professor in the field of "Computational Mathematics". From 1962 to 1979 he was Research Scientist at Computer Center of Moscow State University. Since 1979 E.V.Zakharov is Professor of Faculty of Computational Mathematics and Cybernetics of the Moscow State University. Professor E.V. Zakharov's special field of research activity is numerical methods for solving integral equations. He wrote about 200 papers including 5 monographs.

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