GAME THEORY

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Summary

Game Theory describes human interaction involving conflict, cooperation and competition, the term Interpersonal Decision Theory is synonymous. The term reflects the fact that most essential features of this field are manifested in parlor games. This topic-level treatment covers large parts of the basic concepts and methods and sketches some field of recent applications. The simultaneous occurrence of strategic, stochastic and dynamic phenomena, the fundamental role of epistemic aspects like knowledge and information and the impact of institutional and organizational structures make game theoretic analysis a highly complex task.

In order to deal with various facts of social interaction different forms of strategic or cooperative game models have been developed. The Normal (or Strategic) Form describes the strategic alternatives and the Extensive Form reflects the evolvement of games in time as governed by players' successive decisions during play. In particular, Repeated Games with Incomplete Information describe iterated plays of the same randomly influenced game about which the players receive asymmetric information. The Coalitional Form describes power of coalitions.

Equilibria and solutions represent various approaches to solve games or to describe stable, fair, expected or just likely to payoffs of games.

In mechanism design an imperfectly informed planner with limited enforcement power creates rules of a game that ensure that any potential population of players by playing an equilibrium according to those rules ends up with a socially desired state.

The Equivalence Principle deals with an important application of game theory to large economies, where due to the dominating power of competition distinct solution concepts asymptotically coincide with the Walrasian equilibria.

Recent applications of game theory to evolutionary biology in evolutionary models of social systems and of learning are also briefly sketched.

Finally, results from game theoretic analysis based on perfectly rational players are contrasted with laboratory experiments that have been performed with real, hence at best boundedly rational, players.

A brief assessment of game theory as a part of Operations Research (or vice versa) concludes.

1. Introduction

Game Theory is a mathematical theory of socio-economic phenomena exhibiting human interaction, i.e., conflict and cooperation between decision making individuals (the players). The theory is based on the structural procedures of mathematics and directed towards problems in various fields of applications.

An appropriate synonym is “Multipersonal Decision Theory”. The main paradigms are those of strategic behavior, incomplete information, and mutual anticipation of actions, bargaining power, fairness and equity.

Game Theory approaches the problem of decisions for a group of individuals under uncertainty; it deals with lack of information about the state of the environment, the state of the interpersonal decision process and the state of the opponent’s incentives and abilities. Hence, a probabilistic context is inevitable. The states of nature as well as the strategic behavior of the players involved are generally thought to be randomly influenced.

In addition, the mutual anticipation of opponent’s strategic behavior, the mutual knowledge (See knowledge) about the opponent’s knowledge and the recursive influence of such kind of consideration on the state of knowledge as well as the resulting strategic consequences are modeled; again they are thought to be randomly influenced. This way an idea of “common knowledge” enters the scene.

Also, Game Theory focuses on aspects of cooperation, enforced by legal contract or by long standing experience. It treats problems of fair distribution of resources, acceptable outcomes to joint operations, the representation of bargaining power and coalitional influence, the *a priori* expectation of gains to be achieved from cooperative decisions. The power of coalitions and the resulting influence of individuals, principles of bargaining and axiomatic treatment of solutions, complaints and threats, efficiency and effectiveness, reputation and learning are being discussed on a formal level.

The performance in strategic or cooperative situations (in “the game”) requires an incentive. A version of utility theory is underlying most game theoretical models. This implies that the individuals involved (the players) are capable of expressing preferences with regard to the decisions at stake. Thus, it is required that, for each player, there is preference ordering or a utility function defined on the set of decisions available to *all players.*

Given a player’s incentives, he may have incomplete (and randomly influence) information about the incentives, preferences, or utilities of his opponents. Indeed, Game Theory is capable of describing situations in which players are uncertain about the game they are playing and the opponents they are facing.

Game Theory is also concerned with clarifying the notion of rational behavior. It does not explicitly so, but the concept appears implicitly formulated in various attempts to find a “solution” of a game. Solutions more or less imply that the players achieve benefits by acting rationally on the basis that everyone else behaves rationally as well.

Game Theory basically uses the language of Mathematics; it embraces the analysis of structural relations due to mathematical thinking. Models are formulated in precise definitions, theorems are stated and proved. The mathematical techniques vary through a great range; they involve linear algebra and analysis, measure theory, probability and statistics, stochastic processes and potential theory, partial differential equations, functional analysis, combinatorics, graph theory, optimization and more.

The main fields of application can be found in economics. However, sociology, political sciences, psychology, industrial organization, management science, biology, warfare etc. are all open to the formulation and formal treatment via games.

Within these various fields Game Theory is set to, the formal mathematical treatment contains various degrees of rigor; descriptions of games may be purely verbal and strategic behavior may be treated in a less rigorous framework. Model builders have a tendency to more or less incorporate the methods and the language of their respective field. In this context Game Theory changes its appearance. Economists tend to a version that resembles their way of thinking in the tradition of ideology of certain schools, biologist use the language of evolutionary theory etc. In such a context, mathematical rigor is sacrificed against greater adherence to the methods and dogmas of the particular field.

Historically, Game Theory developed along various different lines of thought, most of them rather disjoint. Mathematicians (in particular the French school LAPLACE, DE MOIVRE, PASCAL) considered the probabilistic aspects of the casino. DANIEL BERNOULLI (1738) (motivated by JEAN and NICOLAS BERNOULLI) considered the St. Petersburg problem; he discussed not only the probabilistic intricacies but also came up with an early version of utility theory. This line was continued by LOUIS BACHELIER (1901), who also created the first version of Brownian motion representing the stock markets fluctuation. EMILE BOREL contributed greatly to put probability theory into its present shape based on measure and integration. (1925-1939). But he was surpassed by JOHN VON NEUMANN when he unsuccessfully tried to solve the Min-Max Problem (1921).

The early economists COURNOT (1838) and BERTRAND (1889) discussed oligopoly and developed a notion of strategy. BERTRAND also treated the game of *baccarat.* This line was continued by EDGEWORTH (1881), ZEUTHEN (1930) and STACKELBERG (1952). EDGEWORTH in addition started a line of discussion leading to the cooperative approach.

At about 1713 (the same time that JEAN AND NICOLAS BERNOULLI report to him the St. Petersburg Problem) DE MONTMORT was also in connection with J. (THE EARL OF) WALDEGRAVE who analyzed a 2-person card game. Here probabilities occur rudimentary reflecting strategic behavior.

Warfare appears in context with strategic thinking. CLAUSEWITZ discusses the battle field coolly from the strategic viewpoint. At the beginning of the 20th century some English engineers developed simple evader-pursuer models which resemble differential games between airplanes.

The decades between 1920 and 1940 reflect the final attempt to view the Theory of Games as a comprehensive field. Von Neumann's proof was based on fixed point theorems which in the mid-thirties where particularly developed by BANACH, MAZUR, ULAM, ERDÖS, STEINHAUS, KURATOWSKI. VILLE was the first to provide a proof based on a separation theorem.

OSKAR MORGENSTERN met VON NEUMANN when both men had to leave Europe in the late 1930s. They laid the foundation of the field of Game Theory with their seminal volume: *The Theory of Games and Economic Behaviour*, in which they stressed the similarity between strategic and cooperative behavior in the economic context as well as in parlor games. Random influence was considered to be inevitable.

Due to these authors three versions of "the game" emerge. Games appear in ***normal form*** (strategic form), in ***extensive form,*** and in ***coalitional form*** (see Foundations of Non-cooperative Game Theory). The first two are close relatives; they constitute the basic paradigm of ***Non-cooperative Game Theory.*** The coalitional form is the basic paradigm of ***Cooperative Game Theory.***

The *normal form* consists of a complete list of possible strategic alternatives for each player. This way each player is assigned a ***strategy space.*** In addition, a ***payoff function*** is specified for each player. Thus, any simultaneous and independent choice of strategies (one be each player) results in a payoff (a real number, a utility, a money term) to each player.

The normal form is also referred to as the ***strategic form,*** in view of the fact that it provides an overview over the strategic options available to a player.

It is one of the main tasks of the model builder to recognize "The Game". Given the data of a multipersonal decision problem of a possibly foggy and unclear nature, one has to specify a normal form game which contains the essential features and is "close" to reality.

For the normal form the basic "solution concept" is ***equilibrium***. This is a strategic situation (an n-tuple of strategies) with dominant stability properties. An equilibrium may reflect versions of "rational behavior" and in some case may be identified with "optimal strategies". In most games, however (as in the real world), there is no "optimal behavior", equilibria may (or may not) exist in abundance and result in gains of greatly varying utility to the players.

The *extensive form* was originally conceived to explain the "rules of the game" (VON NEUMANN MORGENSTERN). Preferably one might think of a time-structured (and stochastically influenced) process that is subject to repeated actions of the players. Intermediate and final payoffs (or costs) are awarded to the players. Decisions at an early state should, therefore, be regarded with respect to the present reward *and* with respect to the future consequences. The process as well as its history may not be fully observable. Players receive private information concerning the state of the process and the choice of actions of the opponents. Strategic behavior is to be defined according to observations and the development of the process. This way, the extensive form results in a normal form and is subject to the analysis theorem. Then equilibrium can be recognized. The extensive form may provide the basic environment, time structure and "rules of the game" but the normal form provides the solution concept

Turning to the *cooperative* or *coalitional form,* we find that the notion of strategy is no more predominant. Rather it is the possibility of *contracts* and *cooperation* which is preeminent. Binding agreements are thought to be possible and enforceable. Thus, the power of coalitions and their influence on the results of a bargaining process is the central topic. A cooperative game is essentially a mapping assigning achievable utilities to coalitions. The task here is to make inference from the power of coalitions to the potential of the individuals. If we now the game, what will be the resulting possibilities, options, expectations, gains to the players?

The "solution concept" of Cooperative Game Theory is the idea of ***Stable Solution.*** While adherent to some idea of equilibrium, the cooperative version of stability is much more static. Stability of the result of bargaining and cooperation, fairness and equity, the returns expected from cooperation, the consequences of an argumentative process, and a final distribution of utility to the players achieved by agreement-these ideas are central to the coalitional form.

Thus, the balance of non-cooperative versus cooperative theory is made precise by discussing strategic behavior and equilibrium strategies versus the power of coalitions and stable solutions.

In the detailed discussion, however, it turns out that the borderline is blurred. There is non-cooperative imitation of cooperation: the stabilizing forces of reputation and punishment that appear in repeated games tend to exhibit elements of cooperation; the agency enforcing contracts can be replaces by the pressure of mutual punishment sustaining equilibrium. On the other hand, cooperative theory incorporates elements of strategic behavior. If uncertainty prevails about the opponent's motivation, their preferences and the game one (thinks one) is involved, the ***mechanisms*** enter the scene. These are devices representing agreements dependent on private observations or knowledge of the players. As these observations cannot be verified independently, players may start to behave strategically with respect to the revelation of their information or their strategies. This sets the individuals involved in a non-cooperative game after the contract had been agreed upon.

Some game theorists hold that cooperative theory is not an "independent" topic, in a sense all cooperation should be explained as resulting from strategic behavior. This view may be extended to a position opposed to cooperative theory at all. Another view, however, is that the idea of the "game" is something Platonic: the paradigm of human competitive and cooperative interaction in the presence of incentives and mutual dependence. Various shapes of this idea materialize, some of them in a precise and mathematically rigorous form.

2. Foundations of Non-cooperative Game Theory

2.1 The Normal Form

The following formal definitions are meant to explain the fundamental topics of Non-cooperative Game Theory.

A non-cooperative n-person game in ***normal form*** is a $2n$-tuple

$$\Gamma = (\mathsf{S}^1,...,\mathsf{S}^n;F^1,...,F^n)\,, \tag{1}$$

with the following ingredients. $\mathsf{S}^i\,(i=1,\ ...\ ,n)$ denotes the set off ***strategies*** of player i**.** This is a complete list of decisions available to the player; at this stage the details of strategic behavior cannot be distinguished. Each F^i is a real valued function defined on the Cartesian product $\mathsf{S} := \mathsf{S}^1 \times ... \times \mathsf{S}^n$ of the strategy spaces. F^i denotes the ***payoff*** to player i, depending on the strategies chosen by *all* players. The choice of a strategy n-tuple is made simultaneously and independently. When preparing his choice each player is not aware of the opponent's intentions. However, communication may take place in advance; a discussion of the merits and demerits of strategy n-tuples may well precede the actual choice of strategies.

A Nash equilibrium (see Foundations of Non-cooperative Theory) is an n-tuple $\bar{s} \in \mathsf{S}$ such that deviating is not profitable for a player provided his opponents stick to their choice. Formally:

$$F^i(\bar{s}) \geq F^i(\bar{s}^1,...,s^i,...,\bar{s}^n),\quad (s^i \in \mathsf{S}^i;\ i=1,...,n)\,. \tag{2}$$

A Priori nothing is said about the establishment of an equilibrium; however, the inherent stability of an equilibrium situation may prevent a player from leaving it. The existence of equilibria requires a basic set of mathematical assumptions, generally the strategy spaces should be (contained in) topological vector spaces and the payoff functions should be quasi-concave and continuous. The standard procedure is to construct the *best reply correspondence* which is a mapping assigning to each $n-1$-tuple of a player's opponents the set of maximizers of his payoff. A fixed point theorem (KAKUTANI, KY FAN) provides a Nash equilibrium. The one first to establish the concept was JOHN F. NASH.

If these conditions fail to apply (e.g., if the strategy sets are finite), then the game may be ***extended*** in various ways. The ***mixed extension*** (see Foundations of Non-cooperative Game Theory) randomizes the strategic choice of strategies. Assume that, on each strategy set S^{i}, there is defined a σ-algebra $\underline{\underline{\mathbf{P}}}^i$ of measurable sets. The probabilities on $\underline{\underline{\mathbf{P}}}^i$ are called ***mixed strategies***. This way player i now chooses a random mechanism which generates his original "pure strategies" (the elements of S^{i})

Given an n-tuple of mixed strategies, the product measure, say $\sigma = \sigma^1 \otimes ... \otimes \sigma^n$ reflects the (stochastically) independent choices of strategies. The expectation $\bar{F}^i(\sigma) = \int F^i(s)\sigma(ds)$ is used to reflect the payoff to player i at this n-tuple of mixed strategies. Now, if $\mathcal{M}^i$ denotes the set of mixed strategies of player i, then we have defined a non-cooperative n-person game in the sense of (1); this is

$$\overline{\Gamma} = (\mathcal{M}^1,...,\mathcal{M}^n;\overline{F}^1,...\overline{F}^n), \tag{3}$$

the *mixed extension* of Γ.

With a suitable structure on the strategy spaces, there is a topology on the mixed strategy spaces (the w^*-*topology*) such that the functions $\overline{F}^i$ are continuous and (multi) –linear with respect to the mixed strategies. This way the above existence theorems can be employed to establish equilibrium in mixed strategies.

Nash equilibrium in mixed strategies can be reinterpreted as follows: for any player i, the $n-1$-tuple of mixed strategies of his opponents are regarded as his ***beliefs*** concerning the behavior of his opponents. A Nash equilibrium constitutes ***consistent beliefs*** of the players concerning their randomized choice of strategies.

The ***correlated extension*** is obtained by introducing a random experiment (a probability space) resulting in private information of the players (i.e., there are subfields of observable events for the players). ***A correlating strategy*** for a player is a random variable, measurable with respect to his observable events and resulting in strategies. If each player chooses such a correlating strategy, the expected payoff for all players (from the composition of the correlating strategies and the payoff functions of the original game) is well defined; hence we have a new normal form, the *correlated extension.* **A correlated equilibrium** *is* a Nash equilibrium of the correlated extension. Actually the mixed extension can be embedded into the correlated one and, for many purposes, suffices to treat the relevant strategic aspects.

There is a host of applications of this model. It is used in oligopolistic competition and other descriptions of price setting mechanisms, statistics, market entry problems, evolutionary biology, for auctions, principle agent problems, inspection problems, insurance contracts, job assignment problems, traffic regulation, etc. in many cases the application of mixed strategies proves to be most successful (see Foundations of Non-cooperative Game Theory).

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