HISTORY OF MATHEMATICS EDUCATION

Kristin Bjarnadottir
University of Iceland – School of Education, Reykjavík, Iceland

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Summary

Mathematics education in the sense of individuals realizing shapes and quantities is as old as civilization. The history is witnessed by clay tablets, papyrus scrolls and books through the centuries. Until the twentieth century, organized mathematics education was generally reserved for young males of higher social strata, usually for their professional training.

The printing technique facilitated access to mathematical knowledge presented in theoretical works as well as popular publications in the vernacular. This process was advanced in the Christian Europe, both by protestant movements and the Catholics. An educational system, organized to provide equality via education for all citizens, was a key to a civic society, the vision of the French Revolution. The nineteenth century saw an increase in elementary schooling for all, including arithmetic for all.

By the twentieth century, professional mathematicians expressed increased concerns about teaching and learning mathematics, especially at secondary school level. The establishment of the international journal *L'Enseignement Mathématique* in 1899 was an important landmark in communicating concerns and ideas, as was the establishment of the International Commission of Mathematics Instruction, ICMI, in 1908.

Interests in the process of learning mathematics encouraged cooperation of mathematics educators with scholars from other disciplines, in particular psychologists, and later with sociologists, anthropologists, etc. The search for means for effective learning of mathematics has lead to reform movements of which the New Math movement of the

1960 is the best known, and to subsequent research. Research areas in the beginning of the twenty-first century are manifold, concerning learning and teaching of mathematics, teacher education, students' attitudes, the life situation of students, the influence of technology on mathematics education, etc. The demand for education for all, including mathematics education for all, has widely been fulfilled, while questions on what to learn, why and how, have yet to be fully answered.

1. Introduction

1.1. Mathematics Education, Its Reasons and Goals

The history of mathematics education is an integral part of the history of mankind's search for knowledge. It concerns the individual's experience of quantities and shapes, it has specific national character in each society and it has developed in international cooperation. Its fundamental reasons are, however, basically homogenous. Niss (1996) has analyzed mathematics education from historical and contemporary perspectives which shows that in essence there are just a few fundamental reasons for mathematics education. They include

- contributing to the *technological and socio-economic development* of society at large, either as such or in competition with other societies/countries;
- contributing to *society's political, ideological and cultural maintenance and development*, again either as such or in competition with other societies/countries;
- *providing individuals with prerequisites which may help them to cope with life* in the various spheres in which they live: education or occupation; private life; social life; life as a citizen.

The *reasons* concern the very existence of mathematics education vis-à-vis given categories of pupils and students, while the term *goal* indicates the actual pursuits of mathematics teaching once it has been established. The goals of mathematics education are often closely related to the underlying reasons for providing it (Niss, 1996).

The early history of mathematics education will be surveyed with respect to the reasons for and goals of teaching and learning mathematics and attitudes towards it, through scholarly texts and textbooks of wide and long-lasting influence. From the turn of the twentieth century, international cooperation within mathematics education and its emergence as a research field will be the main concerns.

1.2. Survey Method and Basic Literature

Many distinguished researchers have written scholarly articles and books about the history of mathematics education and its various aspects. This chapter is based upon the works of many of them.

Gert Schubring, Professor Emeritus at the University of Bielefeld, Germany, has written extensively on the history of mathematics education in his long and productive career. Section 2 is to a large extent based on Schubring's paper on 'mathematics for all' and his writings on reforms of secondary school mathematics by the turn of the twentieth

century. Section 3 on the origin of ICMI, the International Commission on Mathematics Instruction, and the final section, Section 5, with reflections on the present and the future are also based on Schubring's writings.

Fulvia Furinghetti, Professor at the University of Genoa, Italy, has, together with colleagues, studied activities in connection with ICMI and its first journal, *L'Enseignement Mathématique*. Section 3 is largely drawn upon the works of Schubring, Furinghetti and their collaborators.

Mogens Niss, Professor at the Roskilde University, Denmark, has written extensively on various aspects of mathematics education, such as its goals, its competencies, trends and future prospects. Niss's writings guide this chapter on many themes in its introduction as well as in its concluding sections.

Jeremy Kilpatrick, Regents Professor at the University of Georgia, United States, has written on the history of research in mathematics education (Kilpatrick, 1992) in addition to other writings on various themes of the field on which this chapter is drawn. Kilpatrick is one of the editors of a *The Third International Handbook of Mathematics Education* to appear in 2013. The section of this chapter on the history of mathematics education in the various parts of the world is drawn upon a chapter in that book by Alexander Karp, Associate Professor at Columbia University, New York, United States.

Many other researchers have composed works which are referred to in this present chapter; the comprehensive book, *A History of Mathematics, an Introduction*, by Victor J. Katz, professor at the University of the District of Columbia, is a standard reference book on the history of mathematics and early mathematics education in particular. Ubiratan d'Ambrosio, Professor Emeritus at the State University of Campinas, São Paulo, Brazil, is the initiator of the special field termed ethnomathematics, the study of mathematics education by the various cultures around the world. References to the various writers, also those who have not yet been mentioned, will be given as their themes appear. On many occasions, the story is best told by the words of those superior writers on whose works this chapter is drawn. The composer of this chapter is obliged to all of them, while all faults which may appear, would be due to her misconceptions.

2. Early History

Section 2 is to a large extent drawn upon G. Schubring's paper "From the few to the many: Historical perspectives on who should learn mathematics", in *Proceedings of the second international conference on on-going research on the history of Mathematics education*, held in Lisbon, October 2–6, 2011 (Bjarnadóttir, Furinghetti, Matos and Schubring, in print). Other sources are mainly *A history of mathematics, an introduction*, by V. J. Katz, and sources by Frank Swetz (1987; 1992), Van Egmond (1980) and Høyrup (2007).

2.1. Mathematics Education in the Early Antiquity

Elementary mathematics was part of the education system in most ancient civilizations, including ancient Egypt, China, the Vedic society in India before 500 BCE, the ancient

Greece, the Roman Empire, etc. In most cases, formal education was only available to male children with a sufficiently high status, wealth or caste.

The first professional group in history which had to apply mathematical knowledge was that of the scribes. Their activity enabled a rational administration of the societies in Mesopotamia and Egypt. Many writings on mathematics and its methodology date back to 1800 BCE. The emergence of mathematics and of writing are due to the same social process of establishing bookkeeping for the goods delivered by the population in the form of taxes, recorded on clay tablets.

The Rhind Mathematical Papyrus and the Moscow Mathematical Papyrus are famous ancient works on mathematics from Egypt. The Rhind Papyrus, essentially an early textbook for Egyptian students, has been dated to approximately 1650 BCE, but it is thought to be a copy of an even older scroll. Land surveyors in Egypt had to measure anew the areas when inundations of the river Nil had changed the former demarcations as a basis for calculating the respective taxes. Indications exist that an Egyptian scribe, whose specialty had been weights and measures, had to keep his knowledge secret and that he would introduce his eldest son into this art, thus training by apprenticeship (Schubring, 2011).

2.2. Greece and the Roman Empire: the Quadrivium

The roots of mathematics education as a field of didactic activity go back several millennia. In the fifth century BCE, Socrates could use adroit questioning to lead a slave boy to discover that the area of a square on the diagonal of another square is twice that of the smaller square, as related in Plato's *Meno* (Kilpatrick, 1992).

Pure mathematics emerged first in Greek city-states, with a strict epistemological and social separation between practical arithmetic and theoretical mathematics. The first forms of a certain general education became established. For the sons of higher social strata within the free citizens, there was some elementary schooling, including arithmetic, and thereafter the possibility to study with a focus on the rhetoric, or to follow a more philosophical-scientific education. Contrary to the training for the highly valued scribal profession in Mesopotamia and in Egypt, the training of practitioners, like land-surveyors, was left to individual initiative or to organization by the respective professional group (Schubring, 2011).

The most important mathematical text and mathematics textbook of antiquity is the *Elements* of Euclid, written about 300 BCE. It has appeared in more editions than any work other than the Bible. In the *Elements*, Euclid, who lived in Alexandria in the present Egypt, deduced the principles of what is now called Euclidean geometry from a small set of axioms. Euclid's accomplishment was to present them in a single, logically coherent framework, making it easy to use and easy to reference, including a system of rigorous mathematical proofs that remains the basis of mathematics to the present day (Katz, 1993). The *Elements* served as the main textbook for teaching mathematics, especially geometry, from the time of its publication until the late nineteenth or early twentieth century.

In the Roman Empire, basic features of the Greek-Hellenist educational structures were adopted and further developed. By the end of Classical Antiquity, these foci of general education became conceptualized as the seven liberal arts, first defined by Plato (424/423–348/347 BCE); the *trivium* for the rhetorical formation and the *quadrivium* for the selection of those who would continue the four mathematical disciplines: arithmetic, geometry, music/harmony, and astronomy. These liberal arts should constitute the counterpart to the traditionally less valued mechanical arts (Schubring, 2011).

Nicomachus of Gerasa (c. 60–120) was a Neo-Pythagorean and ideas presented in his textbook, *Introduction to Arithmetic* (*Arithmetike eisagoge*), were to exert impact on European arithmetic textbooks into the nineteenth century. Nicomachus's ideas were conveyed to European medieval education by Boëthius (c. 480–524/5) who was a philosopher, born in Rome. His loose translation of Nicomachus's treatise on arithmetic, *De institutione arithmetica libri duo*, contributed to medieval education in mathematics. Nicomachus wrote that unity was not a polygonal number. This meaning became confused in Boëthius's translation with the result that one was believed not to be a number and early modern age authors promoted this idea (Swetz, 1992).

2.3. Middle Ages

China was the first state to introduce official and sophisticated exams for entering its administrative careers. Mathematics was one of the subjects for these exams which became systematically organized by the sixth century to become in practice for about 700 years. There was a well-structured curriculum, with programs and textbooks for each of the exam disciplines. These exams are of particular interest for mathematics learning, since they led to the first official list of textbooks admitted for the preparatory training, comprising among others the *Jiu zhang suan shu* – the *Nine Chapters of Calculation* (Schubring, 2011).

The rise of the Frankish Empire facilitated a certain advancement of learning in the Christian Europe of the Middle Ages. In some schools attached to monasteries, some parts of the seven liberal arts were taught, but in view of the future career of priests; the mathematical knowledge taught there focused on the *computus*, basic astronomical knowledge to calculate the calendar for the religious holidays (Schubring, 2011). The Medieval Latin manuscript *Propositiones ad Acuendos Juvenes – Problems to Sharpen the Young* – is one of the earliest known collections of recreational mathematics problems. The text is attributed to Alcuin of York (735–804) who was an English scholar and teacher from York, Northumbria. At the invitation of Charlemagne, the King of the Franks and Emperor of the Romans, he became a leading scholar and teacher at the Carolingian court. The problems often require some ingenuity to solve but do not depend on any mathematical theory or rule of procedure (Katz, 1993). Manuscripts from this period and later contain many recreational mathematics problems, many of which may have been conveyed from generation to generation and from culture to culture (Tropfke, 1989).

Āryabhaṭa was born in 476 in India. He was the author of several treatises on mathematics, among them his major work, *Āryabhaṭīya*, a compendium of mathematics and astronomy, which has survived to modern times. The place-value system, first seen

in the third century, was clearly in place in his work. The place-value idea, transferred to the Islamic world from India and later to Europe, was to exert important impact on the mathematics education of the world.

The earliest text which deals with Hindu numbers, available in Latin translations, is the *Kitāb al-jam´val tafrīq bi hisāb al-Hind – Book on addition and subtraction after the method of the Indians* by Muhammad ibn-Mūsā al-Khwārizmī (c. 780–850). In his text, al-Khwārizmī introduced nine characters to designate the first nine numbers and a circle to designate zero. He demonstrated how to write any number using these characters in our familiar place-value notation. Al-Khwārizmī's work was important, not only in the Islamic world but also because it introduced many Europeans to the basics of the decimal place-value system (Katz, 1993).

2.4. Place Value Numerical Systems

Interests in mathematics education in the Christian West was further advanced with the works of Gerbert (c. 955–1003), later pope Sylvester II**.** He was born in France but was educated in Catalonia in Spain where he became acquainted with Islamic science. The Muslims had fallen heir to both Greek and Persian science in their initial expansion and had translated many classics into Arabic. At the same time, Arabic traders and travelers were in contact with India and China and had absorbed many of their advances. Gerbert's work represents the first appearance in the Christian West of the Hindu-Arabic numerals, although the absence of the zero and the lack of suitable algorithms for calculating these counters showed that Gerbert did not understand the full significance of the Hindu-Arabic system. During the twelfth century, the cultural exchange among the three major civilizations of Europe and the Mediterranean basin – the Jewish, Christian and Islamic – was very intense. The Islamic supremacy was on the wane and the other two were gaining strength (Katz, 1993). Islamic writings, e.g. by al-Khwārizmī, were eagerly translated from Arabic into Latin in the twelfth century and so was the Greek literature which the Muslims had translated from Greek.

For educational purposes, three works were important in the dissemination process of the decimal place-value system. Alexander de Villa-Dei wrote the hexameter *Carmen de Algorismo*. The poem, which is only a few hundred lines, reached a wide distribution and was even translated into vernacular languages, such as French and the Old Norse language. It summarizes the art of calculating with the new style of *Talibus Indorum*, as Villa-Dei called the new Hindu-Arabic numerals. It is considered to be the first work where the zero appears in European literature. Another important work aimed at use in the medieval cathedral schools was Sacrabosco's *Algorismus Vulgaris*.

The third work is of a different origin. The arithmetic book *Liber Abaci* by Leonardo Bonacci Pisano, often called Fibonacci, is a substantial text on the Hindu-Arabic numerals. The many surviving manuscripts testify to the wide readership the book enjoyed. The book showed the practical importance of the new numeral system, and its calculation methods. It contained applications to commercial bookkeeping, currency conversions as well as conversions of weights and measures, the calculation of interests and other applications. The book was well received throughout educated Europe and had a profound impact on European thought.

While there were channels between the civilizations on the Eurasian continent, there were other civilizations, e.g. in the Western Hemisphere, that developed their own mathematics education which was not realized by others until much later. The Mayan civilization flourished in Mexico and adjacent areas and had its high point between the third and the ninth century. They had a priestly class who studied mathematics. Their numeration system was a mixed system, on one level a place-value system with base twenty. The Inca civilization flourished in what is now Peru and surrounding areas from about 1400 to 1560. They possessed a logical numbering system of recording knots and chords of what is called *quipus* (Katz, 1993). There were more civilizations with their own advanced mathematics, but as these civilizations were destroyed they did not make impact on the global picture of mathematics education.

2.5. The Renaissance and the Abbacus Schools

In the late Middle Ages, from the 13th century on, the first universities began to function in Western Europe. They attracted students, who could afford it, from various regions of Europe to study law, medicine or theology. In the preparatory Faculty of Arts, the seven liberal arts were taught to the youngsters. The lectures of the *quadrivium* were rather marginal, delivered as 'extraordinary' ones, while the *trivium* constituted the core of 'ordinary' lectures (Schubring, 2011).

Occupational opportunities broadened from agricultural and pastoral pursuits to include participation in activities of manufacture and commerce. Apprentices to trades, such as masons, merchants and money-lenders, could expect to learn such practical mathematics as was relevant to their profession. If a student wished to learn commercial arithmetic he usually did not go to a university where arithmetic was taught as one of the subject of the *quadrivium* under the influence of scholasticism, but he sought out a reckoning master, a man skilled in the arts of commercial computation, with whom to study. In Italy they were called *maestri d'abbaco*; and in the German territories; *Rechenmeister*. Many of them accepted students for private tuition or conducted formal group classes in their art which gave rise to reckoning schools, whose number rapidly increased in the commercial cities and along trade routes of Europe (Swetz, 1987). This development originated alongside with the Renaissance in northern Italian cities, such as Florence, Venice, Milan and Bologna. The *scuole d'abbaco*, reckoning schools, emerged, spreading from 14th century Florence, providing training in calculating techniques necessary for the trading commerce (Schubring, 2011).

Since one of the primary areas of trade for the Italian merchants was the Near East, it is natural to assume that they should have taken a special interest in the methods of accounting and mathematics used by the Arabs. The widespread adoption of the Arabic numeral system, which is first noted among the Italian merchants of the thirteenth century, was undoubtedly a result of their close contacts with the Arab world and the new demands created by their increasingly complex system of business organization. The commercial revolution created a need for a new mathematics system to go with the new methods of business organization and the Arabic system filled that need perfectly (Van Egmond, 1980). The earliest attested abbacus master taught in Bologna in 1265, probably on a private basis. Within the next four decades, abbacus masters turned up in numerous other towns from Umbria and Tuscany in the south, to Genoa, Lombardy and

Venice in the north. Masters paid by city communes turned up in sources from 1280s onwards – mainly in smaller communes. Venice and Florence appear to have felt no need for a public undertaking. The Florentine schools were soon considered to be the best, and many Florentines went to teach in other places (Høyrup, 2007). There were six abbacus schools in Florence in 1343 and there were an average of three or four such schools operating continuously in Florence from the earliest decades of the fourteenth century right through the sixteenth and probably beyond (Van Egmond, 1980). In 1613, Nuremberg alone had 48 such institutions (Swetz, 1987).

The normal entrance age in the abbacus schools was ten to eleven years and the normal duration of the training was around two years. At first, students were taught to write numbers in the Arabic number system, followed by the multiplication tables and their applications (Høyrup, 2007). They were taught how to deal with fractions and how to solve basic mathematical problems. Sections of the course were devoted to understanding of the complex Florentine monetary system. The school day followed a familiar routine of lessons, exercises and recitations. Nearly all educated men of the Renaissance gained their basic understanding in a school such as these. When grouped with earlier schools of reading and writing, higher schools of Latin grammar and the educational apex of the university, it is apparent that the abbacus schools were an integral part of a well-designed educational system (Van Egmond, 1980).

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Biographical Sketch

Kristín Bjarnadóttir studied physics and mathematics at the University of Iceland and received her M.Sc.-degree in mathematics at the University of Oregon in 1983. She completed a Ph.D.-degree at Roskilde University, Denmark, in the field of mathematics education in 2006 under the supervision of Prof. Mogens Niss. She taught at secondary schools in Iceland until 2003, when she became assistant professor and later associate professor at Iceland University of Education, and from 2008 at the University of Iceland, School of Education. She has written textbooks for the compulsory school level and an introductory textbook in discrete mathematics for the upper secondary school level. In 1999 she was the chief editor of two national curriculum guides on mathematics in Iceland, one for compulsory school level, and the other for upper secondary school levels. Her research area is the history of mathematics education. She has published a number of scientific articles in that area as well as studies on learning and teaching of mathematics at upper secondary school level.