MATHEMATICAL MODELS IN SPATIAL ECONOMICS

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Summary

This chapter presents a selection of mathematical models in spatial economics, where the geography is represented by one-dimensional as well as two-dimensional (Euclidean) space. The review concentrates on model families for which we provide more detailed examples representing a class of models. A first family of such models deals with the determination of market areas and competition between the few in one-dimensional and two-dimensional space, where firms decide about price and location simultaneously. With reference to the spatial impossibility theorem a lot of attention is given to the importance of pre-located factors for the nature of results. The second family contains equilibrium flow models in two-dimensional space, which depict

location and specialization phenomena, interregional trade and spatial price equilibrium. The third family contains urban land-use models, for which both classical and recent contributions frequently employ a one-dimensional representation of space. Introducing monopolistic competition into such models allows for analyses of externalities and agglomeration phenomena and stresses the importance of interaction costs. The fourth model class relates to the so-called new economic geography and contains examples where each region is modeled as a monocentric one-dimensional space where the global geography consists of a discrete set of such regions. For these latter models the presentation discusses the existence of multiple equilibria, path dependence and the evolution of a hierarchical city system.

1. Introduction

The problem of allocation in space has been approached by economists since the early 19th century. Already in 1826 Johann Heinrich von Thünen formulated a continuous one-dimensional model of optimal allocation of land to different agricultural uses. Although based on highly simplified assumptions of technologies of production and transportation, this model still remains an important basis of much of modern spatial analysis. Examples of modern formulations of land use models built on the one-dimensional representation of space are the models of urban land use by William Alonso (1964), Ed Mills (1967) and Masahisa Fujita (1989).

By the end of the 19th century Wilhelm Launhardt and Alfred Weber introduced the two-dimensional Euclidian space into modeling of optimal location. This important approach to location was later developed into more general models of allocation in space by Tord Palander, August Lösch, Martin Beckmann and Tönu Puu.

In parallel to the development of models in continuous space, analyses of spatial allocation proceeded along the basis of discrete representations of space. This development was inspired by international trade models, as formulated by David Ricardo and Bertil Ohlin. It was a short step to proceed from the two-factor, two-good, two-nation modeling of international trade to multiregional trade and location models, dominating regional economics after World War II.

Time was kept implicit in early mathematical modeling of allocation in space, as in most of standard economic theorizing. The dominating procedure was to apply the principle of comparative statics in the analysis of consequences of changes in some central economic parameter, for instance the level of technology or the supply of capital or labor.

With the increasing interest in economic growth in contributions by e.g. John von Neumann, Roy Harrod and Robert Solow dynamic modeling was applied in the spatial or regional context from the 1960s. From the late 1970s issues of dynamic and structural stability have become important areas of research into regional and spatial economic dynamics. Another new strand of research is models that can predict the existence of urban concentration, in the context of urban structure, inter-urban trade and formation of cities.

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Today regional economics as a discipline is characterized by discrete representations of space, while spatial economics treats economic activities continuously in one- or two-dimensional Euclidian space. It still remains a theoretical challenge to develop a consistent interface between regional and spatial economics. The subsequent presentation touches on this issue, but concentrates on equilibrium patterns of trade flows and on models that can predict the existence of agglomerations and the internal structure of urban areas.

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2. Market Areas and Competition in Continuous Space

2.1. The von Thünen Model and Land Use Planning

Spatial economic models have an inherent element of indeterminacy, which is usually remedied by introducing some fixed element, sometimes referred to as a pre-located factor. The famous von Thünen model from the beginning of the 19th century provides a good illustration of this (von Thünen, 1826). In this model there is a continuous two-dimensional space, in which all produced goods have to be sold in a pre-determined market place. The economic problem is how to locate different activities around the market place, i.e., how to organize the land use. This is a decision problem for the landowners who we may consider as living somewhere else. The solution to the problem is the evolution and establishment of a city or an urban landscape, for which land values are highest at the market place and decrease as the distance to the market place increases.

To illustrate the model we introduce different production activities, indexed $i = 1, 2, .., I$. Decision support is obtained by calculating the profit, $\pi_i(r)$, that activity i can obtain at each distance r from the market place

$$\pi_i(r) = p_i - \tau_i r \tag{1}$$

where p_i is the price minus production costs of good i when sold at the market place, and where τ_i is the transport cost per unit distance for good i. To find out about the best use of each piece of land we can calculate

$$\pi(r) = \underset{\mathrm{i}}{\mathrm{Max}}(p_i - \tau_i r) \tag{2}$$

By repeating the calculation in (2) for all different distances r, it is possible to find the profitability of all land, and the landlord can decide to rent the land at distance r to a producer that satisfies $\pi_i(r) = \pi(r)$, where $\pi_i(r)$ can be thought of as a the bid-rent function for producers of good i across all values of r, and where $\pi(r)$ determines the land rent that land owners will accept at each distance from the market place.

Observe now that – with all prices given – τ_i will determine how fast $\pi_i(r)$ falls as r increases. Moreover, let the production activities be ordered in such a way $p_i > p_{i+1}$ for all activities i. Then the classical land-use result obtains when the following conditions hold simultaneously

$$\tau_i > \tau_{i+1}$$

$$\pi_i(r) > \pi_{i+1}(r) \text{ as } r < \bar{r}$$

$$\pi_i(r) < \pi_{i+1}(r) \text{ as } r > \bar{r}$$

for some distance $\bar{r}$ and all activities i. When the described condition is satisfied, goods with a high price and high distance sensitivity (high τ_i) locate close to the market place, whereas those with a lower price and lower distance sensitivity can find a place farther away from the center. This pattern will only include activities for which there is an r such that net profits are non-negative. The pattern implies optimal land use and the benefits from optimality is reflected by the total land rent collected by the land owners, which is maximized.

2.2. Market Area and Competition

The model in the previous subsection describes producers who compete for land and this results in a location pattern. There is large set of spatial competition models in which the prices are not given and in which each firm's location is a matter of choice. In this case the market area of an individual firm depends on its own and its competitors' location as well as on its own and its competitors' chosen prices.

In the analysis of market areas and spatial competition, the literature often makes the distinction between three forms of price-setting regimes: mill pricing, discriminatory pricing and uniform pricing. To illustrate the first two forms of pricing policies, consider a firm with a linear demand function $p = a/4b - 3v/4$ $q = a - bp$ and variable (marginal) cost of production equal to v. The demand function is assumed to apply to each point in a circular area, populated by customers with the same density everywhere. The profit of a unit sold to a customer at distance r equals $p - v - \tau r$, if the firm pays transport costs and $p - v$ otherwise. Having reached this point we follow Puu's presentation of the case of two-dimensional market areas for the firm (Puu, 2003).

When the firm employs a mill pricing policy the profit, for a given mill price p, is described by

$$\Pi = (p - v)(a - bp)\pi R^2 - b\tau(p - v)2\pi R^3/3 \qquad (3)$$

where R is the radius of a circular market area. This profit expression can be compared with the one that obtains for discriminatory pricing in a circular market area, as specified in (4)

$$\Pi = (p - v - \tau r)(a - bp)\pi R^2 - \tau(p - bp)2\pi R^3/3 \qquad (4)$$

In order to find solutions for each of the two cases we can differentiate profits with respect to p and with respect to R. In this way the mill price comes out as

$$p = a/4b - 3v/4 \qquad (5)$$

which implies that a buyer at distance r has to pay a total price, $p(r) =$ $a/4b - 3v/4$ + τr. This price is evidently higher than $p(r) = v + \tau r$, which is the competitive price.

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Under discriminatory pricing policy the price charged at each distance equals

$$p(r) = a/2b + v/2 + \tau r/2 \qquad (6)$$

and we can se that close to the mill (where r is small), the discriminatory price yields a higher total price than does the buyer's total price associated with mill pricing.

The above outline of market area analysis refers to an area served by one single supplier, which is a case of spatial monopoly. We could see that the size of the market area was a variable in itself, whereas the demand intensity (location of customers) is the pre-determined element. If the same analysis is extended to several suppliers who compete, a whole set of market area compositions can be generated as equilibrium solutions (see, Puu, 2003). Space does not allow further treatment of these interesting results here. Instead we present the classical model of spatial duopoly in a one-dimensional space – a so-called line economy, with the predetermined length l. The variable element is instead how two firms divide this line between them (Figure 1).

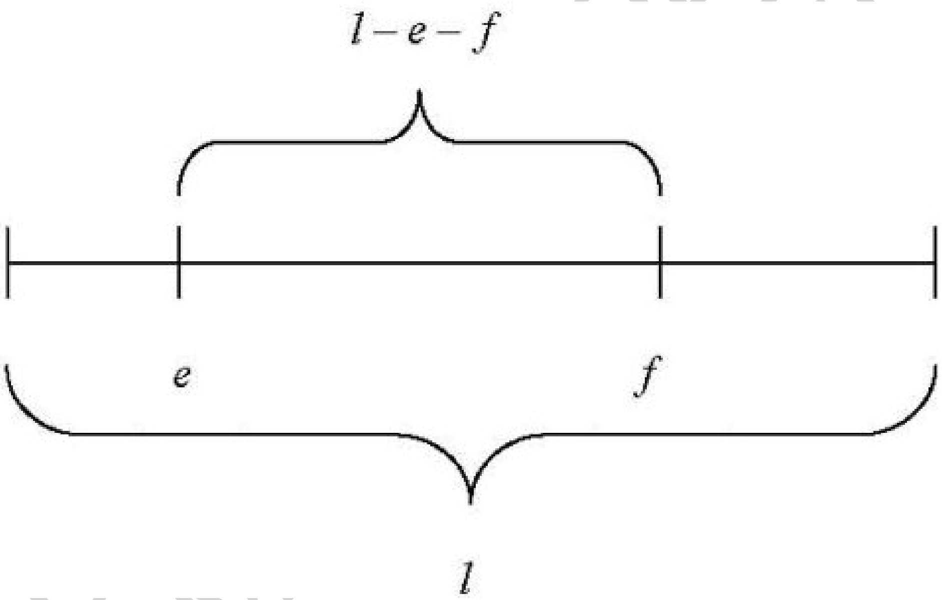

Figure 1. The line economy for a spatial duopoly

Let e and f denote the distance from two endpoints of the line, and assume that one firm is located according to the value of e, while the second firm is located at distance f from the other endpoint. The exposition will follow Beckmann and Thisse (1986) and considers a mill-pricing policy, where p_1 and p_2 denote the mill prices charged by firms 1 and 2. In this version of the model all customers, distributed evenly along the line, demand one unit of the good. Then the demand available to firm 1 can be written as in (7) when the second price equals $\overline{p}_2$:

$$q(p_1, \overline{p}_2) = e + \left[\overline{p}_2 - p_1 + \tau(l - e - f)\right]/2\tau \qquad (7)$$

which applies when $\overline{p}_2 - \tau(l-e-f) \le p_1 \le \overline{p}_2 + \tau(l-e-f)$. This is a requirement that none of the firms has such a low price that it captures the entire market. Moreover, we can see that all demand to the left of point e is captured by firm 1, and all demand to the right of f is captured by firm 2. Whenever a price equilibrium exists it satisfies:

$$p_1^* = v + \tau\left[l + (e-f)/3\right] \text{ and } p_2^* = v + \tau\left[l + (f-e)/3\right] \qquad (8)$$

where v denotes variable cost per unit produced. This equilibrium exists iff

$$l + (e-f)/3 \ge 4l(e+2f)/3 \text{ and } l + (f-e)/3 \ge 4l(f+2e)/3 \qquad (9)$$

The model version sketched above is usually referred to as the Hotelling spatial duopoly (Hotelling, 1929). Puu (2002) provides an overview of different specifications and corresponding solutions to the model when demand is elastic. As pointed out by Beckmann and Thisse (1986), the Hotelling model is a two-agent non-cooperative game, and the solution described is a Nash equilibrium. The Hotelling model and related spatial oligopoly models have been researched extensively for a lot of variations in assumptions and market set up, and generating several different market area tessellations for two-dimensional representation of space. (Puu, 2003, Puu and Sushko, 2002).

2.3. Spatial Impossibility Theorem

A basic feature of a competitive equilibrium is that firms are price takers. As shown by Starret (1978), such an equilibrium cannot be established for an economy operating in a homogenous space, unless total transport costs are equal to zero, which is basically equivalent to not having any space. The problem is that a competitive price system is unable to govern trade flows on the one hand, and support a profit-maximizing location of each firm on the other hand.

If we want to analyze city formation, spatial specialization and trade, we can either find ways to model space as heterogeneous or introduce some form of imperfect competition, based on technological externalities or monopolistic competition, where each firm has its own negatively sloping demand curve (Fujita, Krugman and Venables, 1999). In the von Thünen model, space is not homogenous because of the central market place in which all transactions take place. On the other hand, in the Weber model (Puu, 2003) as well as in the model of spatial price equilibrium (Beckmann, 1952; Samuelson, 1952), certain resources or activities are pre-located.

In Beckman and Puu (1985) we can observe excess supply and excess demand at different points in two-dimensional space, and for such situations equilibrium trade flows can be derived. In their long-run equilibrium analysis, the authors assume that households may own firms in other locations than their place of residence, and they conclude that without such an established structure “no interregional trade could occur, since trade would balance everywhere”. This observation comes close to a re-statement of the observation made by Starret.

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