MECHANICS OF SOLIDS

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Summary

The subject matter is described together with the notion of solids and the central concepts of stress and strain. A historical note on the development of solid mechanics follows. General considerations on the continuum hypothesis, the theoretical background and the elastic modulii, precede comments around various issues dealing with applications, electromagnetic effects and thermodynamic aspects. The Classical Theory of Elasticity is then described through short discussions on the problem of beams and plates, body and surface waves, and the conceptual and mathematical foundation of the Navier equation. The essay ends with brief comments on fracture, finite elasticity, computational mechanics and granular materials.

1. Introduction

This subject is a well-developed area of Mechanics, a major division of Physics, and studies the static and dynamic behaviour of solid materials. At first hand it seems obvious what a solid is, as opposed to a liquid or a gas, such as water and air, but it is a matter of time scales what might be considered a solid. A fluid, the generic term that encompasses liquids and gases, is a material that moves indefinitely when a force is applied parallel to its surface, a shear force, like the wind blowing over a body of water. In contrast, a solid will deform up to some limit when a shear force acts on it, provided it doesn't exceed the yield limit or fractures the material. Glass, at standard temperature and pressure, appears quite solid on the time scales of seconds and months, while it slowly flows or creeps down under the effect of the earth's gravitational field, in the scale of hundreds of years, as it is apparent in the stained glass windows in medieval gothic cathedrals; at higher temperatures this *plastic* behaviour becomes more evident.

Usually, this area is viewed as a part of what is called Classical Physics, in the sense that Newton's laws of motion are fully applicable and no concern is given to the constituents of matter, atoms and molecules, where Quantum Mechanics provides the appropriate conceptual framework. Furthermore, large masses or velocities, such as the realm of galactic structures or a spinning neutron star, are also excluded, as Relativistic considerations must be taken into account.

The standard approach is to assume that a solid is a piece of material that maintains its identity as forces are applied to it; it is made up of infinitely many parts. This is the so-called continuum hypothesis. Under this assumption, the theory addresses the way in which a solid deforms under a given load distribution. That is, its goal is to predict the response of a solid material when the forces acting on it are known. The latter, when divided by the area on which they act are called stresses; those parallel to the surface are shear stresses, and the normal or perpendicular forces per unit area are known as normal stresses. Any material, solid or fluid can withstand a normal stress. The resulting deformation of a solid under specified stresses is formulated in terms of a displacement field, which characterizes how each point of the material is displaced as a result of the load at its original position. It contains the size of the displacement and the direction in which it takes place; it is a vector quantity. As this vector field changes from point to point, it is important to know its spatial variation. This is given by the strain field, a tensor quantity, which can be viewed as the local elongation per unit length. One might imagine a vector field as an infinite array of small arrows, at each point in space, that specifies a local direction and some number related to the size of the arrow; the velocity (vector) field in a flow provides the direction and magnitude of the velocity of the fluid at each point. A tensor field, a richer entity, gives some property, at each point, in an infinite number of directions; the strain (tensor) field includes, at every point, the local displacement and how it varies in all possible directions.

Solids can be characterized by their response under a given load. Consider a rectangular solid sample in between two parallel flat plates glued to opposing faces. If the plates are pushed together compressing the sample, a normal stress deforms the material; likewise if the plates are displaced parallel to each other in opposite directions, a shear stress will also lead to a deformation. In either case, when the plates are set back to their initial position the sample rearranges itself; if it fully recovers its shape, the material is said to be *elastic*; it is called plastic if some permanent deformation remains. Again, this behaviour will depend on the conditions under which this loading test is performed. At low temperatures, short times and small enough loads, most materials are elastic. As these quantities increase, *plastic* behaviour is observed unless the material is brittle and fractures. *Viscoelastic* substances are those that display a mixed behaviour, exhibiting both elastic and creeping features. These materials, along with those showing more complicated responses are the subject study of Rheology, which is somewhere in between Fluid and Solid Mechanics, sharing the same basic principles but dealing with complicated responses.

2. Historical Notes

The practical handling of solid materials goes back, of course, to the development of tools, weapons and shelters in prehistoric times. The Renaissance brought light to the earliest quantitative attempts to determine tensile strength and the concept of stress, mainly due to Leonardo Da Vinci and, a hundred years later, Galileo Galilei. By 1660, Robert Hooke had discovered the important result that bears his name, first published as an anagram (*ut tensio sic vis*) in 1676, which translated freely states that the deformation is proportional to the applied force; this is Hooke's law. Isaac Newton's monumental work establishing the basic principles of Mechanics set the stage to all future developments. With these cornerstones, the privileged minds of Leonhard Euler and members of the Bernoulli family, mainly Jakob, Johann, and Daniel, began formulating the appropriate mathematical language of continuum mechanics and addressing the solution of the first elastic problems during the eighteenth century. In this period Charles-Augustine Coulomb started to relate the somewhat formal mathematical results to actual problems. The next century, mainly due to the French school represented by Claude-Louis-Marie Navier, Augustine-Louis Cauchy, and Siméon-Denis Poisson, saw the general formulation of the Theory of Elasticity almost in its present way; in fact, this was done for the case of small (infinitesimal) deformations. Major contributions were also made by George Green and Thomas Young (Lord Kelvin), introducing the recently developed concept of energy and the Principles of Thermodynamics. Though many names come to mind when following the conceptual and practical evolution of the field, besides those mentioned above, credit must be given to Adhémar-Jean-Claude Barré de Saint Venant for a principle that bears his name and significantly simplifies the practical use of the theory. The formulation of a theory for finite displacements required another century to be addressed and the advent of digital computers to be fully accessible.

3. General Considerations

In order to be able to predict the dynamic behaviour of any solid material when the forces acting on it are known, a set of equations describing the spatial and temporal variation of the displacement field must be formulated, together with the appropriate initial and boundary conditions. The theory stems from Newton's laws of motion for the case of a continuum; Euler was able to do so in one of his most famous memoirs. The next key ingredient was the concept of stress, which was later introduced by Cauchy. With these elements, and a particular model for the "molecules" and their mutual interactions, Navier accomplished the task of giving the theory a structure that still remains. Using appropriate averaging procedures to "smooth out" the macroscopic details, he developed a phenomenological theory in which the collective behaviour of the constitutive particles defines the elastic response through the elastic modulii. Of course, many important details required years of research to be fully understood, modified or incorporated to the general theory.

In its present form, the theory is summarized by the equations relating the product of the density of the material and the local acceleration, to the forces per unit area acting at that point. The latter are divided into a part that describes the local stresses, due to the surrounding material, and a part that "models" the external environment, such as gravitational or electromagnetic forces. It is in this first part where the specific elastic response of each material becomes apparent. This is done by relating the stresses to the strains at each point, the so-called constitutive or stress-strain relations. If the mutual dependence is in direct proportion, the proportionality factor being the elastic parameters (elastic tensor), the equations are linear. It was Cauchy who first and clearly showed that the elastic tensor could be expressed as an array of 81 numbers, of which only a subset of 36, as a result of symmetry considerations, were independent of each other. A controversy aroused as to what were the maximum and minimum numbers of elastic modulii that a given material required to be completely characterized. It was Green who proved that for the most anisotropic material it suffices to determine 21 constants, while for the simplest possible case only two were needed; this was obtained by constructing an expression for the elastic energy, a novel approach at the time since thermodynamics was in its earliest stages of development.

For a completely isotropic and homogeneous material, in which no preferred directions or positions are present, two elastic parameters are sufficient to fully characterize the solid. These are known as the Lamé constants (λ and μ), or the Poisson ratio (ν) and Young's modulus (E), or the shear (G) and bulk (K) modulii. Depending on which form is suitable for a particular problem, any two parameters may be used, as well defined relations exist among them, v. g.: $E\nu=\lambda(1\text{-}2\nu)(1+\nu)$, or $3K(1\text{-}2\nu)=E$. Physically, the set E-ν is simpler to interpret. For example, suppose that a cylindrical rod is subject to a uniform pull at the ends (uniaxial tensile stress). The rod will extend in this direction and will contract along the two mutually perpendicular directions in the circular cross section as it becomes thinner. In this case, ν represents the ratio of transverse contraction and longitudinal extension, while E is the ratio of the applied longitudinal force and the resulting extension. The first is dimensionless, as it is the ratio of two lengths (the lateral contraction and the longitudinal extension of the rod), and E has the dimensions of stress or pressure (force/length^2), typically the pascal (1 Pa = 1 N/m^2) or the dyne/cm^2. The values of ν are between 0 and ½; for example, on average, 0.17 for glass, 0.27 for structural steel, 0.33 for aluminium alloys, and 0.48 for rubbers. For E some illustrative average values (in 10^{10} Pa) are 6.9 for aluminium alloys and glasses, 19 for stainless steel, 69 for tungsten carbide, and 0.35 for polystyrene.

Once the theory was formulated, mathematicians, physicists, and engineers could readily address a variety of problems that led to many areas of research that are yet to be completed. The early studies of extension, compression, and flexure of beams, going back to Galileo and passing through the able hands of Euler, the Bernoullis, and Coulomb were then extended to three dimensional analyses and given a sound basis that could be useful for various technological applications, mainly oriented to structural and stability calculations. Also, problems dealing with torsion of different solid configurations, strained plates and shells, solid contacts and impact, fracture and strength of materials, and the generation and propagation of waves in a solid material were, and still are, an active field of research in the most diverse directions and with the richest motivations.

When the solid can respond electrically or magnetically, the theory must be coupled with James Clerk Maxwell's equations describing all electromagnetic interactions. This is the case of conducting metals in the presence of magnetic fields, as in coils, and electric transformers. Situations in which stresses give rise to magnetic or electric effects, as in piezoelectric crystals, are also a case in point; these effects are known as magneto- or electro-striction. Thermal considerations may also be important in the stability of a structure or in its elastic response. For example, the density of a material and its elastic modulii depend on temperature; it is common knowledge that variations of the latter produce contraction or expansion of a solid. The first law of thermodynamics, stating energy conservation, takes into account that besides the mechanical work performed by the internal stresses or the external forces, the energy content of any material may be altered due to heat flow. Additionally, the second law of thermodynamics establishes restrictions on possible stress-strain relations, inequalities that various elastic properties must satisfy, as well as the accessible states a system can attain when allowed to evolve by removing imposed initial conditions, as in the removal of a mechanical constraint under thermal isolation (adiabatic process) or with an isothermal restriction by careful manipulation of heat sources.

In most instances, time-dependent problems in solid mechanics deal with either the stability of a static (equilibrium) structure or on the propagation of waves in a material. Otherwise, seldom is there any concern with the time dependence of the various strain or stress fields. That is, once the initial state is known, including the structural and geometrical features of the sample, and the stress distribution, the goal is to determine the final state, regardless of how this is reached; this accomplished, the stability of the final state is analysed. This means introducing some small perturbation on the solution, like slightly changing the displacement field or the imposed stresses, and following its time evolution; if this variation grows indefinitely in time, the structure is said to be unstable. The other case in which time is of any concern is that in which the material is forced externally at some initial time, producing a disturbance that propagates through it. The results are transient and steady waves of various types.

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Biographical Sketch

Ramon Peralta Fabi, was born in México City, México in 1948. Graduate and undergraduate studies were at the Universidad Nacional Autónoma de México (UNAM): Physics degree 1971, M. Sc. in 1973, D. Sc. in 1975. The doctoral thesis was developed at the University of Maryland and the California Institute of Technology, in the US, under the supervision of Prof. Robert Zwanzig. Professor of Physics at the Universidad Autónoma Metropolitana-Iztapalapa from 1975 to 1983, Professor of Physics since 1983 at the UNAM. Alexander von Humboldt Research Fellow 1979. Vice-president (1980-1982) and President (1983-1984) of the Mexican Physical Society. President of the Division of Fluid Dynamcs (1993-1995). Member of the National Sytem of Research since 1983. Distinguished as "Catedrático-UNAM" since 1998. Taught over 75 courses in Physics and Mathematics, at undergraduate and graduate levels, supervised more than 25 theses, written over 50 publications, including specialized papers, a book for the general public and essays on various scientific issues. Main areas of research are Fluid Dynamics and Statistical Physics; lately interested in the Physics of Granular Materials.