This chapter illustrates the application of continuum mechanics to the modeling of solid materials through the development of specific constitutive equations adapted to each material. A general view of the most used and useful approaches and constitutive theories applicable to the deformation, fatigue and fracture of metals, composite and biological materials are reviewed.

Deformation of metals and alloys has been customarily modeled by plastic theories based on yield criteria, with damage and failure dealt with through independent criteria. Fracture Mechanics concepts, introduced at the last half of the 20th century, have helped to integrate failure and fatigue theories and are the basis of new developments.

Composite materials have become standard in structural application in engineering because their outstanding specific properties (stiffness and strength) and the possibility of tailoring the microstructure to obtain a given set of properties. The relationship between the microstructural characteristics (volume fraction, shape and spatial distribution of matrix and reinforcement) and the macroscopic behavior can be obtained by means of homogenization methods. They were initially developed for the elastic regime and have been extended in recent years to deal with plasticity and damage.

Biological materials show a striking combination of optimized properties such as strength, toughness and compliance. Due to the fact that soft biological materials show a highly nonlinear, incompressible behavior, and that they are ordinarily subjected to
large strains under a complex multiaxial stress state, the initial models drawn from polymer science have given way to phenomenological constitutive equations with a more or less close connection to the microscopical constituents.

1. Introduction

This chapter illustrates the application of continuum mechanics to the modeling of solid materials through the development of specific constitutive equations adapted to each material. The following sections summarize some of the most important constitutive theories applicable to the deformation fatigue and fracture of metals, composite and biological materials aiming at giving to the reader a general view of the most used and useful approaches.

It has not been the intention of the authors to give a detailed and thorough description of all the available models and theories but to show in a simple and intelligible way the methods and techniques used to deal with involved topics like damage, fracture or heterogeneity within the framework of the continuum mechanics. All the sections of this chapter are self-contained and can be read independently. The authors presume that the reader has a basic knowledge of continuum mechanics, and that it is acquainted with the tensor notation.

2. Application to Metals

Metals and alloys are materials being used by the human being since the end of the Stone Age. However, the scientific knowledge of its mechanical behavior begins at the XIX Century. The theories to model plastic behavior of metals and alloys were developed along the XX Century. Tresca (1864) and Von Mises (1913) proposed their yield criteria. Levy (1864), Mises (1913), Prandtl (1924) and Reuss (1930) established the stress-strain relationship. Failure criteria were proposed from the pioneering work of Hancock and Mackenzie (1976); finally, fatigue behavior was modeled using Fracture Mechanics concepts since the work of Paris and Erdogan (1963). This section summarizes the State of the Art of Solid Mechanics applications to metals and alloys.

2.1. Plasticity Models

For metals and alloys, the plastic strain rate tensor $\dot{\varepsilon}^p$ is a function of the deviatoric stress tensor $\sigma'$

$$\dot{\varepsilon}^p = f (\sigma')$$  \hspace{1cm} (1)

After some mathematical manipulation, Prandtl-Reuss equations are derived

$$\dot{\varepsilon}^p = f (\sigma') = \frac{3\dot{\sigma}}{2\sigma^2 F'} \sigma'$$  \hspace{1cm} (2)

where $\sigma$ is the equivalent stress (Mises stress) defined as
\[
\bar{\sigma} = \sqrt{\frac{3}{2}} \sigma \cdot \sigma'
\]  

(3)

\( F' \) is the derivative of function defined as the relationship between \( \bar{\sigma} \) and the plastic work per unit volume

\[
\bar{\sigma} = F \left( \int \sigma' \cdot d\varepsilon^p \right)
\]  

(4)

This equation is taking into account the increase of yielding stress with plastic work (work hardening). Alternatively, it can be substituted by the following expression, more artificial although easier to use

\[
\bar{\sigma} = H \left( \int d\varepsilon^p \right)
\]  

(5)

where \( \dot{\varepsilon}^p \) is the rate of equivalent plastic strain, defined as

\[
\dot{\varepsilon}^p = \frac{2}{\sqrt{3}} \varepsilon^p \cdot \hat{\varepsilon}^p
\]  

(6)

and Prandtl-Reuss equations yield

\[
\dot{\varepsilon}^p = \frac{3 \hat{\sigma}}{2 \sigma H'} \sigma'
\]  

(7)

valid for \( \bar{\sigma} > 0 \) (loading), otherwise \( \dot{\varepsilon}^p = 0 \) (unloading).

\( H (\bullet) \) is thus the function relating equivalent stress and equivalent plastic strain. For uniaxial stress conditions (tension or compression), \( \bar{\sigma} \) coincides with the applied stress, and \( \varepsilon^p \) coincides with plastic strain in the direction of the applied stress, so that function \( H (\bullet) \) is the stress-plastic strain curve obtained in a tensile test with the alloy.

Such stress-plastic strain curve used to be dependent on temperature and strain rate. The most widely used analytical formulae are those by Johnson-Cook and Zerilli-Armstrong. Johnson-Cook equation is an empirical formula

\[
\sigma = \left[ A + B \left( \varepsilon^p \right)^n \right] \left( 1 + C \ln \dot{\varepsilon}^* \right) \left( 1 - T'^m \right)
\]  

(8)

where \( \dot{\varepsilon}^* \) is the strain rate ratio

\[
\dot{\varepsilon}^* = \frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}
\]  

(9)
\( \dot{\varepsilon} \) being the actual strain rate, \( \dot{\varepsilon}_0 \) a reference strain rate and \( T^* \) is a temperature factor given by

\[
T^* = \frac{T - T_0}{T_m - T_0}
\]

(10)

where \( T \) is actual temperature, \( T_0 \) a reference temperature and \( T_m \) the melting temperature. A, B, C, m and n are empirical constants.

Zerilli-Armstrong expressions are derived from dislocation motion equations. For FCC metals

\[
\sigma = C_0 + C_2 \left( \dot{\varepsilon}^n \right)^{\frac{1}{p}} \exp \left\{ -C_4 T + C_4 T \ln \dot{\varepsilon} \right\}
\]

(11)

For BCC metals

\[
\sigma = C_0 + C_1 \exp \left\{ -C_4 T + C_4 T \ln \dot{\varepsilon} \right\} + C_5 \left( \dot{\varepsilon}^n \right)^{\frac{1}{p}}
\]

(12)

where \( \dot{\varepsilon} \) is the strain rate, \( T \) the absolute temperature and \( C_0, C_1, C_2, C_3, C_4, C_5 \) and \( n \) are material constants to be determined experimentally.

**2.2. Damage and Fracture Models**

Continuum Mechanics models of ductile fracture of metals and alloys are usually developed by transforming the general equations of Plasticity to incorporate into the most common damage mechanisms. These consist of the nucleation, growth and coalescence of voids from second phase particles that fail by fracture or decohesion from the metal matrix. The metal or alloy is identified with a porous material whose porosity increases with plastic deformation due to continuous nucleation of new voids and growth of the old ones. Void coalescence is the damage process determinant of failure modes such as ductile crack growth or localized plastic flow and shear fracture. The most widely known porous ductile material model is originally due to Gurson [2.1], even though posterior contributions have fruitfully improved it (see, for instance [2.2]). A scalar field representing the void fraction is incorporated into the constitutive equations of Plasticity to account for the influence of porosity on the macroscopic deformation. As a counterbalance, a porosity growth law must be added to the constitutive equations. Since plastic strain does not change volume, this law must express that the porosity growth rate only differs from the macroscopic volume strain rate in the contribution of void nucleation to the porosity rate.

The constitutive equations modified by the incorporation of porosity are the yield condition and the stress-strain relations. Further, the spherical part of the stress tensor also must incorporated into these set of equations, since this component of stresses largely influences porosity, even though the plastic strains of the metal matrix do not depend on it.
Accordingly, a very simple model of porous ductile material can be formulated by assuming a rigid, perfectly plastic metal matrix, in which void nucleation and yielding occur simultaneously and no further void nucleation takes place. For such a material, plastic strains are the complete strains, the yield condition of the matrix becomes reduced to the substitution of the function $F(\bullet)$ of Eq (4) by a material constant, the yield strength $\sigma_Y$, and the porosity growth rate coincides with the macroscopic volume strain rate. According to the original Gurson model, the yield condition, the stress-strain relations, and the porosity growth law for this porous material are given in the next table and compared with the analogous equations for the same material in the absence of porosity.

<table>
<thead>
<tr>
<th></th>
<th>Non porous material</th>
<th>Porous material</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Yield condition</strong></td>
<td>$\sigma = \sigma_Y$</td>
<td>$\bar{\sigma} = \sigma_Y \sqrt{1 + \nu^2 - 2\nu \cosh \left( \frac{3\sigma_m}{2\sigma_Y} \right)}$ (13)</td>
</tr>
<tr>
<td><strong>Stress-strain</strong></td>
<td>$\dot{\varepsilon} = \dot{\varepsilon}_p = \dot{\lambda} \frac{3}{\sigma_Y} \sigma'$</td>
<td>$\dot{\varepsilon} = \dot{\varepsilon}_p = \dot{\lambda} \left[ \frac{3}{\sigma_Y} \sigma' + \nu \frac{\sigma_m}{\sigma_Y} \right] \sinh \left( \frac{3\sigma_m}{2\sigma_Y} \right) \mathbf{1}$ (14)</td>
</tr>
<tr>
<td><strong>Porosity growth</strong></td>
<td>$\text{tr}\dot{\varepsilon} = \text{tr}\dot{\varepsilon}_p = 0$</td>
<td>$\text{tr}\dot{\varepsilon} = \text{tr}\dot{\varepsilon}_p = \frac{\nu}{1 - \nu}$ (15)</td>
</tr>
</tbody>
</table>

In these formulae, $\bar{\sigma}$ is the von Mises stress, $\sigma_m$ the spherical stress, $\nu$ the void fraction, $\sigma'$ is the deviator stress tensor, $\dot{\varepsilon}$ and $\dot{\varepsilon}_p$ are the complete and plastic strain-rate tensors, $\dot{\lambda}$ is an indeterminate proportionality factor, $\mathbf{1}$ is the identity tensor, and a superimposed dot denotes time derivative. Plasticity problems involving a porous material as described by the equations of the third column of the table are solved by using these equations together with the general Continuum Mechanics ones (equilibrium, compatibility).

For porous materials with isotropic work-hardening matrix, the yield strength $\sigma_Y$ of the matrix is not a material constant, but a function $\sigma_Y = F(w_p)$ of plastic work per unit volume $w_p$, as stated in the previous section. To keep the balance between unknowns and equations, the condition of equal plastic work rate in the matrix and the porous material must be added. This yields the additional equation

$$\sigma \cdot \dot{\varepsilon}_p = (1 - \nu) \dot{w}_p = (1 - \nu) f'(\sigma_Y) \dot{\sigma}_Y$$ (16)

$f'(\cdot)$ being the derivative of the inverse function $f(\cdot)$ of $F(\cdot)$. 

©Encyclopedia of Life Support Systems (EOLSS)
Bibliography


Biographical Sketches

**Manuel Elices** (MSc. Civil Engineering (1963), MSc. Physics (1964), PhD. (1966)) is Emeritus Professor of Materials Science and Technology at the Polytechnic University of Madrid. Professor Elices' professional and research work has always centred upon Materials Science. He has been Associate Editor of many major international journals in the field (Acta Materialia and Scripta Materialia among others) and he has served of Chairman, Counsellor or Member of the Advisory Committee of various international organisations (ACI, FIP, RILEM). Prof. Elices has carried out research on the mechanical behaviour of materials in both the experimental and theoretical fields. As guest lecturer he has given courses in a number of universities in Europe, the U.S.A., South America, Japan and China. He introduced in Spain the new engineering degree of Materials Science and Engineering, and he has published more than 300 scientific papers, 10 books, and contributed chapters in another 10 books. Professor Elices is a Life Member of the Real Academia de Ciencias Exactas, Fisicas y Naturales and of the Academia de Ingenieria de España. Member of the European Academy (Materials Science Section), Honorary Doctor of the Universities of Navarra and of Carlos III and the holder of a number of National and International prizes (Bengough, Metals Society, DuPont, Honorary Member of ESIS, Spanish National Prize of Technological Research Leonardo Torres Quevedo, Real Academia de Ciencias Medal, Polytechnic University of Madrid Medal and Ramon LLull among others).

**Vicente Sanchez-Galvez** is Civil Engineer from Polytechnic University of Madrid, UPM, (1971), Physicist from Complutensis University (1975) and Doctor from UPM (1975). He is Full Professor at the Civil Engineering School of UPM (1983). He is author of nine books and over 200 papers on mechanical properties of materials. He has been Director of the Civil Engineering School (1989-1997) and Vicerector of UPM (2000-2007). Currently he is the Director of the Materials Science Department of UPM as well as Director of The Research Centre for Safety and Durability of Structures and Materials (CISDEM).

**Andrés Valiente** (MSc. Civil Engineering (1975), PhD. (1980), MSc. Physics (1982) ) is Professor of Materials Science at the Polytechnic University of Madrid. His professional and research work has always centred upon mechanical performance of structural materials and structural integrity.

**Prof. Javier Llorca** graduated in Civil Engineering at the Polytechnic University of Madrid in 1983. He got his Ph. D. in Materials Science at the same institution in 1986 and was appointed Associate Professor in the Department of Materials Science in 1987 and Professor in 1995. He is currently head of the research group on “Advanced Structural Materials and Nanomaterials” at the Polytechnic University of Madrid and Director of Madrid Institute for Advanced Studies of Materials (IMDEA Materials). His research activity has been focused in the analysis of the relationship between microstructure and mechanical properties in advanced structural materials. Prof. L.Llorca has held positions as invited professor at Brown and Cornell University and has received various awards, including the Research Award from the Spanish Royal Academy of Sciences, the Research award from the Polytechnic University of Madrid and the Gold Medal from the Spanish Structural Integrity Society. He is currently Associate Editor of Modelling and Simulation in Materials Science and Engineering, Composites Science
and Technology, International Journal of Fatigue and Fracture of Engineering Materials and Structures, and International Journal of Engineering Sciences. In the framework of his research activities, he has co-authored over 130 research papers in international peer-reviewed journals, holds an h index of 30, and has given about one hundred invited talks at national and international conferences and research centers throughout the world.

**Gustavo V. Guinea** (Madrid, Spain; 1962) is professor of materials science at the Universidad Politécnica de Madrid, where he obtained his M.Sc. and Ph.D. degrees in Civil Engineering in 1986 and 1990, respectively. He also holds a M.Sc. in Physics from the Universidad Complutense de Madrid where he graduated in 1988. Among other prizes, he was awarded the RILEM's Robert L'Hermite Medal in 1993, and in 2006 he was appointed to Correspondent Member of the Royal Spanish Academy of Sciences. Prof. Guinea is since 2002 the President of the Spanish Structural Integrity Society. Expert on mechanical behavior and fracture of structural materials, he has a relevant research experience in characterization and modeling of mechanical properties and microstructure of biological fibers and soft tissues.