IMPLICIT CONSTITUTIVE RELATIONS

K. R. Rajagopal
Texas A&M University, USA

Keywords: Constitutive Equations, Implicit models, Implicit Theories, Solid, Fluids

Contents

1. Introduction
2. A Simple Implicit Model
3. More General Implicit Theories for Fluids
4. Implicit Theories for Elastic Solids
5. More General Implicit Theories
6. Concluding Remarks
Glossary
Bibliography
Biographical Sketch

Summary

Classical Constitutive equations like the Navier-Stokes model for fluids and the linearized elastic model for solids are explicit models in that they provide explicit expressions for the stress in terms of kinematical quantities. Similarly, the constitutive equation for a Simple Material is also an explicit equation for the stress in terms of the histories of the deformation gradient and the density. However, many rate type models that have been developed to describe viscoelastic and inelastic materials are implicit in that a equation is provided for the stress, and its time rates, as well as appropriate kinematical quantities and their time rates. Even when time rates are not involved, such implicit models are very useful in describing a large class of materials, especially those wherein the material moduli depend on the Lagrange multiplier that is associated with a constraint. Thus, for example, such implicit equations can describe incompressible materials in which the material moduli depend on the pressure (mean normal stress). In this chapter we discuss very briefly the role of implicit constitutive relations in mechanics.

1. Introduction

Classical constitutive relations such as those that are used to describe the linearized elastic and linear viscoelastic (To be precise, the model should also be referred to as the linearized viscoelastic model as the strain that is used in the constitutive representation is the linearized strain rather than a fully and proper non-linear strain measure.) response of solids, namely Hooke's law (see Hooke's Potentia de Resistuva in Gunther (1931)) and the linear viscoelastic solid model (Boltzmann (1874)), as well as the Euler fluid model (Euler (1752), (1755) and the Navier-Stokes (Navier (1823), Poisson (1831), St. Venant (1843), Stokes (1845)) models for compressible and incompressible fluids are all explicit constitutive relations in the sense that they provide an explicit expression for the stress. In the case of the classical linearized elastic model and the linear viscoelastic solid models one can also provide explicit expressions for the linearized strain in terms of the stress (Though the classical Navier-Stokes model is not described by providing an expression for the symmetric part of the velocity gradient in terms of the stress, one can do so.). Many, but not all, of the popular models that are in vogue fall under the category of constitutive expressions wherein the stress can be expressed as a functional of the history of the deformation gradient, a category introduced by Noll (1954) which he called simple materials. However, models for plasticity and many of the rate type theories that are being used do not fall into the category of simple materials introduced by Noll (1957), (1958). The rate type model defined by Truesdell and Noll (1992) is a special type of rate model in that the pth order derivative of the stress is related to the stress, and its (p-1) time derivatives as well as the first r derivatives of the deformation gradient. However, they realized that even this special rate type model they define need not belong to the class of simple fluids as they remark "In general it is not possible to reconstruct from the relation (36.2) the corresponding constitutive equation (36.1). In fact, it is conceivable that a single relation of the form (36.2) be satisfied for several different simple materials, and that some solutions not correspond to some materials at all. Thus we must regard a particular differential equation (36.2) as defining a class, possibly empty, of materials of the rate type rather than a single such material". More general rate type models definitely do not fall into the class of simple fluids. Noll (1972) later generalized his definition of a simple material to include rate type models for plasticity, etc. However, even this generalization retains the feature of expressing the stress as a functional of kinematical quantities and some *ad hoc* state variables which do not have clear physical underpinning though there is a discussion of some thermodynamic issues which are far from complete. In any event, such models do not belong to the general implicit constitutive theories we shall discuss. The point is Noll's theories always define the stress in terms of functionals of certain kinematical variable. They do not include for instance a theory in which all one has is a relation between the stress and its various derivatives as well as kinematical quantities and their various derivatives, related implicitly. For instance, it is not possible to describe certain types of material response, say that for constrained materials whose material properties depend on the Lagrange multiplier that enforces the constraint (It would be appropriate at this juncture to point to the fact that most models that have been proposed to describe the inelastic response of materials are rate type models which are not simple materials in the sense of Noll's original definition in 1958. However, while his new theory of simple materials could be used to model inelastic response, there is absolutely no discussion of how such a theory can be put into place to describe the response of a specific material or a discussion of the physical underpinnings) . We shall discuss this issue in detail later.

It is worth pointing out that constitutive models have been proposed wherein the material moduli, that describe the body, for instance, depend on the mean normal stress as well as the shear stresses in the body leading to implicit models for the stress and the symmetric part of the velocity gradient. Saal and Koens (1933) assumed that the viscosity of asphaltic bitumen depended on both the shear stress and normal stress, i.e., they had a truly implicit constitutive theory, and Bingham and Stephens (1934) investigated the effect of pressure on the "fluidity" of bodies (see Murali Krishnan and Rajagopal (2003) for a discussion of the relevant issues). More recently, Morgan (1966) has discussed the use of implicit constitutive theories, but the scope of his work is limited.

Let us for a moment consider a very simple ideal model. Suppose we have a generalization of the classical incompressible Navier-Stokes fluid whose viscosity depends on both the pressure and the symmetric part of the velocity gradient. There is nothing with regard to physics which proscribes the consideration of such a model and we will see that the stress in such a body is expressed in terms of the mean value of the stress and kinematical variables; the constitutive expression is necessarily an implicit relation.

There is unfortunately a cavalier attitude when it comes to describing how a body is constituted. One tends to use the terms constitutive relation and constitutive equation interchangeably but nothing could be more inappropriate. The first implies that there is a relation between various quantities like the stress, strain, rate of strain, etc., which could be an implicit relationship, while the second implies that one of the quantities can be expressed as a function of the others.

Many implicit rate type constitutive models have been introduced to describe the behavior of solids and fluids. However some early rate type constitutive theories like those introduced by Maxwell (1866) to describe the behavior of viscoelastic fluids can also be integrated and the stress can be expressed explicitly in terms of the history of kinematical quantities. However, not all rate type models can be integrated to yield explicit expressions for the stress in terms of the history of the deformation gradient. In fact, this is one of the early erroneous conclusions in plasticity theory wherein the stress is assumed to depend on the strain and the "plastic strain" and a rate equation is given for the plastic strain. If the rate equation for the plastic strain could be integrated to provide an expression for it in terms of the history of the deformation gradient, then one would in fact have an explicit expression for the stress and we would have a simple material in the sense of Noll (1957). The problem is that invariably such an integration for determining the "plastic strain" cannot be carried out.

2. A Simple Implicit Model

That the viscosity in a fluid can depend upon the pressure was recognized several centuries ago. A detailed discussion of the history of experiments concerning the dependence of viscosity on pressure (Here one ought to be careful to recognize that in all the instances mentioned, the "pressure" that is referred to is the "pressure" of the fluid in the confining medium and that the "pressure" in the fluid that is being tested is assumed to be uniform throughout the fluid that is being tested and equals the pressure in the confining medium.) can be found in the book by Bridgman (1931). The early experiments of Perkins involved dropping a Cannon into the depths of the Ocean in order to create a high pressure environment for the fluid being tested which filled the Cannon, and the inventive work of Amagat, wherein "*Amagat developed a special packing technique by which he was able to consistently reach pressures of 3000 kg./cm.*^{2} *or more.*" (see Bridgman (1935). In the section on how the viscosity of fluids varies with pressure Bridgman refers to the early experiments of Roentgen, Warburg and Sachs, Hauser, Cohen, and other investigators all of whom used some form of a capillary flow method for a number of lubricating oils, and found the viscosity of the test fluid rising with pressure.

Stokes (1845) also recognized the fact that the viscosity of a fluid depends on the pressure and states "If we suppose μ to be independent of the pressure also, and substitute . . ." the implication here is that in general the viscosity could depend on the pressure but that he is making such an assumption in a particular class of flows. In fact, Stokes' intent becomes clear when he comments soon afterward in the paper "Let us now consider in what cases it is allowable to suppose μ to be independent of the pressure. It has been concluded by Du Buat from his experiments on the motion of water in pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure . . . I shall therefore suppose that for water, and by analogy for other incompressible fluids, μ is independent of the pressure". He does not explicitly state whether he is referring to the "mechanical pressure" or the "thermodynamic pressure".

Barus (1893) proposed the following exponential relationship between viscosity and pressure:

$$\mu = \mu_0 \exp(\alpha p), \tag{1}$$

where α has units $(Pa)^{-1}$ and p is expressed in (Pa). That viscosity in fact rises sharply with pressure has been repeatedly verified (see experiments cited in Hron et al. (2003) and see Bair & Koptte 2003, figure 1). Using Barus' equation to get a rough estimate of the variation in the viscosity with pressure for Naphthalemic mineral oil α has been determined experimentally to be 26.5GPa^{-1} at 20 ◦C, 23.4GPa^{-1} at40 ◦C, 20GPa^{-1} at 60 ◦C and 16.4GPa^{-1} at 80 ◦C (see Hogland 1999). Thus a change of pressure from 0.1GPa to 1.0 GPa at 60 ◦C leads to a change in the viscosity of 4.85 × 108 %! The density on the other hand changes according to the relation (see Dowson-Higginson 1966)

$$\rho = \rho_0 \left[1 + \frac{(0.6)\,p}{1 + (1.7)\,p} \right]. \tag{2}$$

Thus, the change in density is approximately 16%. For a smaller range in the variation of pressure, while the viscosity can vary significantly, the density variation could be merely a few percent (the percentage change in the density when the pressure changes from 2 to 3GPa is approximately 3.5 %). While such a change in density needs to be taken into account if one is interested in describing the response accurately, in most applications one can ignore the density change and model the fluid as being incompressible.

Andrade as quoted in Bridgman's book (see also Andrade (1930)) suggested the following dependence of the viscosity on pressure, density and temperature:

$$\mu(p,\rho,\theta) = A\rho^{1/2}\exp\left[\left(p+\rho r^2\right)\frac{s}{\theta}\right], \tag{3}$$

where ρ denotes the density, θ the temperature, p the pressure, and r, s and A are constants. Here once again the pressure refers to the pressure of the confining fluid rather than the confined fluid that is tested, but the tacit assumption is that the pressure in the fluid being tested is uniform and that of the confining fluid as remarked earlier. However, since the variation in the density is small and since the density can be treated as a constant in such cases, the viscosity depends on the pressure in the fluid and the temperature.

It is well known that several fluids have the ability to shear thin as well as shear thicken. Let us confine our attention to the class of fluids that are incompressible with the stress given by the "relation"

$$\mathbf{T} = -p\mathbf{1} + \left[\hat{\mu}\left(p, 1\mathbf{D}1^2, \theta\right)\right]\mathbf{D}, \tag{4}$$

where $\mathbf{D}$ is the symmetric part of the velocity gradient.

Since the fluid is in incompressible, it can undergo only isochoric motions and thus

$$\mathrm{tr}\mathbf{D} = 0, \tag{5}$$

which implies that

$$p = -\frac{1}{3}\mathrm{tr}\mathbf{T}. \tag{6}$$

Thus, it is reasonable when subject to a wide range of pressures, the viscosity could depend on both the pressure and the shear rate (symmetric part of the velocity gradient).

This would lead to a model of the following kind:

$$\mathbf{T} = -\left(\frac{1}{3}\mathrm{tr}\mathbf{T}\right)\mathbf{1} + \left(\hat{\mu}\left[\mathrm{tr}\mathbf{T}, 1\mathbf{D}1^2\right], \theta\right)\mathbf{D}. \tag{7}$$

where $\hat{\mu}$ is a function of the first invariant of the stress and the second invariant of the symmetric part of the velocity gradient. Thus, depending on the form of the viscosity, one might not be able to express the stress explicitly in terms of the symmetric part of the velocity gradient. This is probably one of the simplest instances of an implicit constitutive relation whose generalization is best expressed as:

$$\mathbf{f}\left(\mathbf{T}, \mathbf{D}, \theta\right) = 0. \tag{8}$$

We notice that (8) is not an explicit relation (and neither is (7)) for the stress as a function of $\mathbf{D}$ but it is an implicit relation.

The Cauchy stress $\mathbf{T}$ in a compressible Navier–Stokes fluid is related to the symmetric part of the velocity gradient through

$$\mathbf{T} = -p(\rho;\theta)\mathbf{1} + \lambda(tr\mathbf{D})\mathbf{1} + 2\mu\mathbf{D} \tag{9}$$

In general, the pressure p (given by an equation of state) and the material moduli λ and μ will depend on the density and the temperature. There is nothing to prevent a generalization of the same, wherein one considers a fluid whose constitutive relation is given by

$$\mathbf{f}(\rho, \mathbf{T}, \mathbf{D}, \theta) = \mathbf{0}, \tag{10}$$

which would be the implicit counterpart for a compressible fluid of the incompressible fluid model (8). Thus, in such a model one allows for the possibility of the "thermodynamic pressure" being different from the mean normal stress.

-
-
-

Bibliography

A. J. A. Morgan, (1966) Some properties of media by constitutive equations in implicit form, *International Journal of Engineering Science*, **4**, 155–178.

B. Bernstein and K. R. Rajagopal, (2008) Thermodynamics of Hypoelasticity, *Zeitschrift fur Angewandte Mathematik und Physik*, **59**, 537-553.

B. Bernstein, (1960) Hypo-elasticity and elasticity, *Arch. Rational Mecn. Anal*. 2, 89–104.

Bernstein, B., (1983)A unified thermodynamic theory of elasticity and plasticity, *Journal of Elasticity*, 13, 411–427.

C. Barus, (1893) Isotherms, isopiestics and isometrics relative to viscosity. *Am. J. Sci*. 45, 87–96.

C. L. M. H. Navier, (1823) Mémoire sur les lois du mouvement des uides. *Mém. Acad. Re. Sci*., Paris, **6**, 389-416.

C. Truesdell and H. Moon, (1975) Inequalities sufficient to ensure semi-invertibility of isotropic functions, *Journal of Elasticity*, 5, 183-189.

C. Truesdell and W.Noll, (1992) *The Non-Linear Field Theories of Mechanics*, 2nd ed. Springer-Verlag, Berlin.

C. Truesdell, (1977) *A First Course in Rational Continuum Mechanics*, Academic Press,.

C. Truesdell, Hypoelasticity, (1955) *J. Rational Mech. Anal.* **4**, 83–133.

C.Truesdell, A Program towards Rediscovering the Rational Mechanics of the Age of Reason, *Archive for the History of Exact Sciences* **1**, 1–34 (1960).

D. Dowson and G. R. Higginson, (1966) *Elastohydrodynamic Lubrication*, *The Fundamentals of Roller and Gear Lubrication*. Pergamon Press.

E. C. Andrade, (1930). Viscosity of liquids. *Nature* 125, 309–310.

E. C. Bingham and Stephens, R. A., (1934) The alternating stress method for the measurement of the consistency of very stiff materials. *Physics* 5, 217–220.

G. F. Gauss, (1830) On a new general principle of mechanics. *Phil. Mag*. 8, 137–140. [Translation of Ueber ein neues allegemeines Grundgesetz der Mechanik. J. Reine Angew. Math. 4, 232–235 (1829).]

I. Newton, (1995) *Philosophiae naturalis principia mathematica* (1687), (translated by Andrew Motte) Prometheus Books, New York.

J. C. Maxwell, (1866) On the dynamical theory of gases, *Phil. Trans. Roy. Soc. Lond*., A157, pp. 26-78.

J. Hron, J. Malek, and K. R. Rajagopal, (2001) Simple flows of fluids with pressure dependent viscosities. *Proc. R. Soc. Lond*. A 457, 1603–1622.

J. L. Lagrange, (1777) *Mechanique analytique*. Mme Ve Courcier, Paris.

J. Murali Krishnan and K.R. Rajagopal, (2003) Review of the uses and modeling of bitumen from ancient to modern times, *Applied Mechanics Reviews*, **56**, 149-214.

K. R. Rajagopal and A. R. Srininvasa, (2004)On the thermomechanics of materials that have multiple natural configurations, part2: Twinning and Solid to Solid phase transitions, *Zeitschrift fur Angewandte Mathematik und Mechanik*, **55**, 1074-1093.

K. R. Rajagopal and A. R. Srinivasa, (2000) Thermodynamic Framework for Rate type constitutive equations, *Journal of Non-Newtonian Fluid Mechanics*, **88**, 207-228.

K. R. Rajagopal and A. R. Srinivasa, (2004) On the thermomechanics of materials that have multiple natural configurations, part 1: Viscoelasticity and Classical Plasticity, *Zeitschrift fur Angewandte Mathematik und Mechanik*, **55**, 861-893.

K. R. Rajagopal and A. R. Srinivasa, (2004) On the thermomechanical restrictions of continua, *Proc. R. Soc. London* A Mat., **460**, 631-651.

K. R. Rajagopal and A. R. Srinivasa, (2007) On the dissipative response of solids, *Proc. Roy. Soc*., London, **463** : 357-367.

K. R. Rajagopal and A. R. Srinivasa, (2007) On the thermodynamics of fluids defined by implicit constitutive relations, *Zeitschrift fur Angewandte Mathematik und Mechanik.*

K. R. Rajagopal and A. S. Wineman, (1980) On Constitutive Equations for Branching of Response with Selectivity, *International Journal of Non-Linear Mechanics*, **15**:83-91.

K. R. Rajagopal, (2003) On Implicit Constitutive Theories, *Applications of Mathematics*, **28**, 279-319.

K. R. Rajagopal, (2006) Implicit Constitutive Theories for Fluids, *Journal of Fluid Mechanics*, **550**, 243–249.

K. R. Rajagopal, Elasticity of Elasticity, (2007) *Zeitschrift fur Angewandte Mathematik und Mechanik*, **58**, 309–317.

L. Euler, (1755) Principes generauz du movement des fluids, *Histoire de L' Academie Royal des Sciences et Belles Lettres*, 274-315.

L. Euler, (1756-1757) *Principia motus fluidorum, Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, **6**, 271-312.

L.Boltzmann, (1874) Sitzungsber. Kaiserlich Akad. Wiss. (Wien) Math. Natuwiss. Classe, 70 (II), 275.

Olsen, E. T and Bernstein, B., (1984) A class of hypo-elastic non-elastic materials and their thermodynamics, *Arch. Rational Mech. Anal*. 86, 291–303.

P. W. Bridgman, (1931) *The Physics of High Pressure*. MacMillan.

R. Hooke, (1931) De Potentia Resistutiva, in R. T. Gunther, *Early Science in Oxford*, Vol.VIII: The Cutler Lectures of Robert Hooke, Oxford: Oxford University Press, pp. 331-356.

R. T. Gunther, (1931) *Early Science in Oxford*, Vol.VIII: The Cutler Lectures of Robert Hooke, Oxford University Press, Oxford, 331-356.

R.N.J. Saal and G. Koens, (1933) Investigation into the plastic properties of asphaltic bitumen, *Physics*, **7**, 408-412.

R.N.J. Saal and J.W.A. Labout, (1958) *Rheological properties of asphalts, Rheology: Theory and Applications*, Vol. II (ed. F.R. Eirich), Academic Press, New York, 363-400.

S. Bair and P. Kottke, (2003). Pressure-viscosity relationships for elastohydrodynamics. *Tribology Trans*. 46, 289–295

S. D. Poisson, (1831) Mémoire sur les équations générales de l'équilibre et du mouvement des corps solides élastiques et des uides. *Journal de l'Ecole Polytechnique*, **13**, 1-174.

Saint Venant, (1843) Not a joindre au Memoire sur la dynamique des fluids, *Comptes Rendus*, **17**, 1240-1224.

W. Noll, (1957) *On the foundations of mechanics of continuous media*, Carnegie Institute of Technology, Department of Mathematics Report 17.

W.Noll, (1972) *A new theory of simple materials*, Archive for Rational Mechanics and Analysis.

Biographical Sketch

Prof. Rajagopal is currently a Distinguished Professor and a Regents Professor at Texas A&M University. He holds the Forsyth Chair in Mechanical Engineering and has joint appointments in the Departments of Mathematics, Biomedical Engineering, Chemical Engineering and Civil Engineering. He is also a Senior Research Scientist at The Texas Transportation Institute. Before moving to Texas A&M University, he was at the University of Pittsburgh where he rose from the rank of assistant professor to professor and held the James T. MacLeod Chair. He also held joint appointments in the Department of Mathematics and the Department of Surgery.Prof. Rajagopal received his undergraduate degree in Mechanical Engineering from the Indian Institute of Technology, his M.S. from the Department of Aerospace Engineering and Mechanical Engineering at the Illinois Institute of Technology and his Ph.D in Aerospace Engineering and Mechanics from the University of Minnesota. He held a post-doctoral lecturership at the University of Michigan, Ann Arbor and then held an assistant professorship at The Catholic University of America.