STRUCTURAL DYNAMICS AND MODAL ANALYSIS

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Contents

Summary

This contribution is devoted to two inter-related topics in the field of Structural Mechanics, namely, Structural Dynamics and Modal Analysis. It has been conceived aiming at providing the reader with the knowledge about the essentials of numerical and experimental techniques developed for characterizing the dynamic behavior of structural systems. In this context, "structural systems" broadly encompass a large range of engineered products such as civil engineering structures (buildings, towers, bridges, etc.), vehicles (airplanes, automobiles, trains, ships, spaceships, etc.), industrial equipment (pipes, off-shore platforms, electric power lines, etc.) and machines (compressors, turbines, internal combustion engines, etc.) and household appliances (refrigerators, washing machines, air-conditioners, etc.).

Structural dynamics is the subject of continuously increasing interest in engineering, especially due to the fact that, in many applications, dynamic behavior is influential upon operational effectiveness, comfort and safety of the products mentioned above. It should not be forgotten that environmental and energy-related aspects, which have become subject of concern, are frequently in relation with structural dynamics

Given the broadness of the topics addressed, this presentation is considered to be introductory. To learn about more advanced aspects, the reader should refer to a number of excellent text-books available, some of which are included in the bibliography list provided.

This chapter comprises five sections, which are devoted to the following aspects of structural dynamics: Section 1: Introduction; Section 2: Vibrations of discrete systems; Section 3: Vibrations of continuous systems; Section 4: Finite element modeling in structural dynamics; Section 5: Introduction to experimental modal analysis.

1. Introduction

In the context of the present contribution, Structural Dynamics, is related to the study of the vibratory behavior of mechanical systems (machines, vehicles, industrial equipment and civil engineering structures, etc.) when subjected either to sustained or impulsive external forces, which are known as excitations. Vibrations are considered as oscillations about the equilibrium position of the system and result from a continuous exchange between kinetic and potential energy. Kinetic energy is related to the system's mass or inertia, while the potential energy is associated to the system's flexibility.

Modal Analysis is understood as the ensemble of analytical and experimental techniques intended for the modeling of the dynamic behavior of vibrating systems that derive from the fact that, under certain conditions, the dynamic response can be represented as a superposition of the dynamic responses of elementary mechanical systems, in terms of the so-called modal characteristics. From the mathematical standpoint, modal analysis can be interpreted as a set of techniques intended for solving partial linear differential equations or systems of linear ordinary differential equations by performing transformations from the physical coordinates to the so-called modal space.

Modal analysis encompasses two main types of techniques, which will be addressed in the remainder of this chapter, namely:

Analytical modal analysis, which is primarily related to the modeling of the dynamic behavior in the modal space, making use of the modal characteristics. Hence, analytical modal analysis is intended to solve a class of direct problems;

Experimental modal analysis, which consists in determining a set of modal characteristics of a given structural system from a set of measured responses. Thus, experimental modal analysis deals with inverse or identification problems.

2. Theoretical Foundations of Structural Dynamics

In the section, the theoretical foundations of structural dynamics and modal analysis are presented under the assumption of linear behavior.

2.1. Equations of Motion for Discrete Systems

We consider herein structural systems modeled as those depicted in Figures 1 and 2, which are assumed to exhibit linear behavior. This means that the resilient elements (springs) establish proportionality between displacements and restoring forces: $f_{\mathrm{e}} = -kx$ and the viscous dashpots (dampers) establish proportionality between velocities and damping forces: $f_{\mathrm{d}} = -c\dot{x}$.

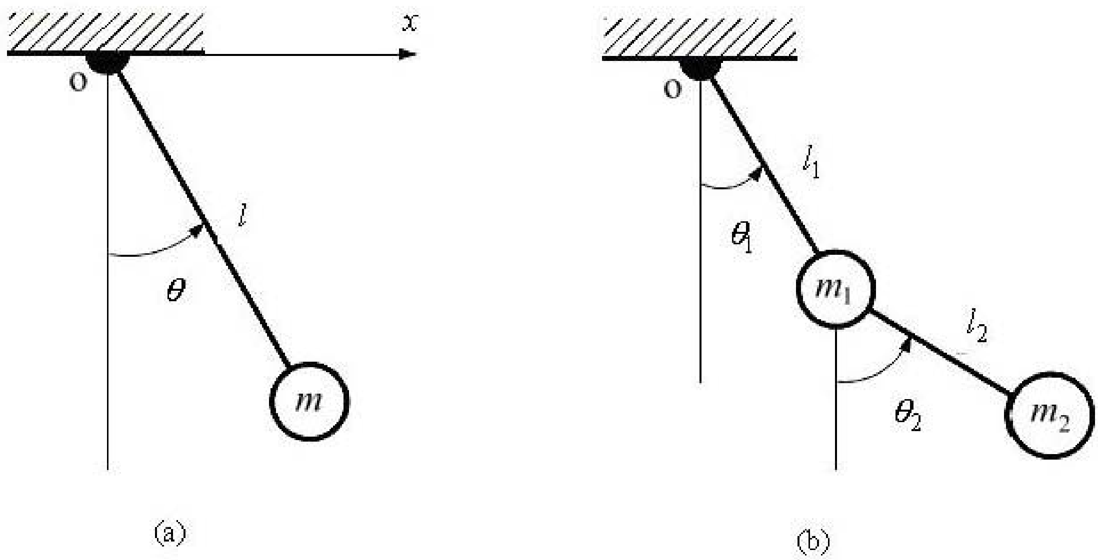

Figure 1. Examples of single d.o.f. and two d.o.f. undamped vibrating systems.

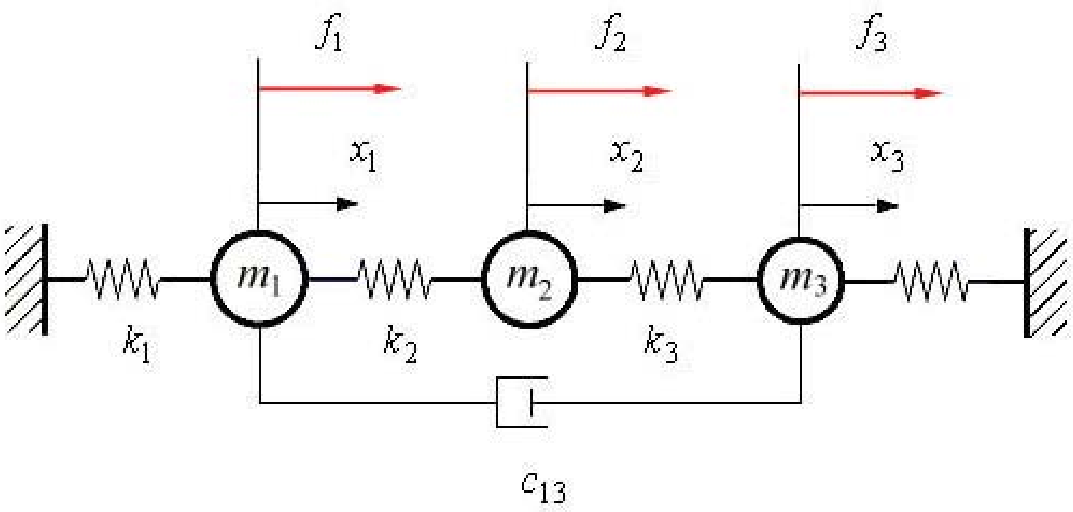

Figure 2. Example of a three d.o.f. damped vibrating system.

From Newton's Second Law or, alternatively, from the so-called Lagrange's equations, it is possible to obtain the set of coupled second-order differential equations of motion that represent the dynamic behavior of a viscously damped N d.o.f. mechanical system of, in the following matrix form:

$$[M]\{\ddot{x}(t)\}+[C]\{\dot{x}(t)\}+[K]\{x(t)\}=\{f(t)\} \tag{2.1}$$

where:

$\{x(t)\}=\begin{Bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{Bmatrix}\in R^n$ is the vector of time responses;

$\{f(t)\}=\begin{Bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{Bmatrix}\in R^n$ is the vector of excitation forces;

Matrices $[M]$ (positive-definite), $[C]$ and $[K]\in R^{n,n}$ (positive definite or semi-positive definite) are the symmetric matrices of mass, damping and stiffness, respectively.

2.2. Undamped Free-Vibrations. Eigenvalues and Eigenvectors

For the undamped system, without any excitation force, the equations of motion given by Eq. (2.1) become:

$$[M]\{\ddot{x}(t)\}+[K]\{x(t)\}=\{0\}, \quad (2.2)$$

whose general solution is searched in the form:

$$x(t)=\{x\}e^{i\omega t} \quad (2.3)$$

This type of solution means that we are assuming harmonic free responses with circular frequency ω.
By introducing Eq. (2.3) into Eq. (2.2), the following *eigenvalue problem* is obtained:

$$([K]-\lambda[M])\{x\}=\{0\} \quad (2.4)$$

where $\lambda=\omega^2$.

The problem expressed by Eq. (2.4) admits N pairs of non-trivial solutions $(\lambda_r,\{x_r\})$, $r=1,2,\ldots,n$, the so-called *eigensolutions*, where:

- $\lambda_r \in R^+$ are the *eigenvalues*. The natural frequencies are given by: $\omega_r=\sqrt{\lambda_r}$, $r=1,2,\ldots,n$;

- $\{x_r\}\in R^n$ are the *eigenvectors* or vibration natural mode-shapes.

The eigenvalues are obtained from the following condition:

$$\det([K]-\lambda[M])=0 \quad (2.5)$$

whose development leads to the following *characteristic polynomial*:

$$\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_0=0 \quad (2.6)$$

For each one of the n eigenvalues, Eq. (2.4) gives the corresponding eigenvector$\{X_r\}$. It is usual to regroup the eigensolutions in the following matrices:

Modal Matrix: $[X]=[\{x_1\} \quad \{x_2\} \quad \cdots \quad \{x_n\}]\in R^{n,n}$ (2.7)

Spectral Matrix: $[\Lambda]=\mathrm{diag}\{\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n\}\in R^{n,n}$ (2.8)

In general, it is assumed that the diagonal entries of the spectral matrix and the columns in the modal matrix are ordered according to the increasing magnitudes of the eigenvalues λ_r.

Matrix $[K]$ can be either positive-definite or positive semi-definite, according to the boundary conditions (kinematic constraints) of the system: when the constraints are sufficient to prevent any motion without deformation of the flexible elements $[K]$ will be positive-definite; on the contrary, $[K]$ will be positive semi-definite. In the first case, all the eigenvalues of $[K]$ will be positive and the corresponding vibration modes will be named as *elastic modes*. In the second case, the system will have a number p of null eigenvalues $(p \leq 6)$, to which correspond the so-called *rigid body modes* and a set of $(n-p)$ elastic modes that are associated with positive eigenvalues.

Examples: The 3 d.o.f. system represented by Figure 3(a) presents a null eigenvalue that corresponds to the rigid body mode represented k_1 by the set of displacements $x_1 = x_2 = x_3$ that may occur without deformation of the springs and k_2. On the other hand, the system represented by Figure 3(b) does not exhibit any null eigenvalue because any arbitrary set of displacements $\{x_1, x_2, x_3\}$ satisfying the boundary conditions leads to the deformation of at least one of the elastic elements.

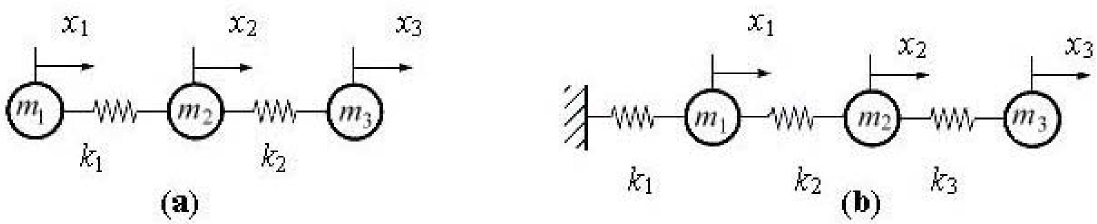

Figure 3. Three d.o.f. system under two different boundary conditions

One of the most important property of the eigenvectors, is the so-called Orthogonality Property, with respect to mass and stiffness matrices, which are demonstrated in the following.

For a given pair of eigensolutions $(\lambda_r, \{x_r\})$, Eq. (2.4) is rewritten as:

$$[K]\{X_r\} = \lambda_r [M]\{X_r\} \tag{2.9}$$

For another pair $(\lambda_s, \{x_s\})$, one has:

$$[K]\{X_s\} = \lambda_s [M]\{X_s\} \tag{2.10}$$

Pre-multiplying Eq. (2.9) by $\{x_s\}^{\mathrm{T}}$ and Eq. (2.10) by $\{x_r\}^{\mathrm{T}}$, successively, subtracting one resulting equation from the other, and accounting for the symmetry of matrices

$[\mathrm{K}]$ and $[\mathrm{M}]$, one writes:

$$\left(\lambda_s - \lambda_r\right)\left\{x_r\right\}^{\mathrm{T}}\left[M\right]\left\{x_s\right\} = 0 \tag{2.11}$$

Assuming distinct eigenvalues, namely $\lambda_r \neq \lambda_s$, Eq. (2.15) is satisfied only if:

$$\left\{x_r\right\}^{\mathrm{T}}\left[M\right]\left\{x_s\right\} = 0 \tag{2.12}$$

$$\left\{x_r\right\}^{\mathrm{T}}\left[K\right]\left\{x_s\right\} = 0, \tag{2.13}$$

for each and every pair of eigenvectors with $r \neq s$.

Altogether, Eqs. (2.12) and (2.13) express the *orthogonality properties* of the eigenvectors, with respect to the mass and stiffness matrices, respectively. This is a fundamental property that is the basis of many of the vibration analysis methods, as will be seen in the remainder.

From Eq. (2.4) it is possible to see that if $\left\{x_r\right\}$ is an eigenvector, any collinear vector $\alpha\left\{x_r\right\}$, with $\alpha \neq 0$ is also an eigenvector. This means that the norms of the eigenvectors are not determined uniquely and can be chosen arbitrarily, so that they satisfy:

$$\left.\begin{aligned}\left\{x_r\right\}^{\mathrm{T}}\left[M\right]\left\{x_r\right\} &= \eta_r \\ \left\{x_r\right\}^{\mathrm{T}}\left[K\right]\left\{x_r\right\} &= \eta_r \lambda_r\end{aligned}\right\}, \quad r = 1, 2, \ldots, n \tag{2.14}$$

where η_r, $r = 1, 2, \ldots, n$ are the so-called *generalized masses*, whose values must be arbitrarily prescribed. Usually the eigenvectors are normalized in such a way to have unit generalized masses, i.e.:

$$\left.\begin{aligned}\left\{x_r\right\}^{\mathrm{T}}\left[M\right]\left\{x_r\right\} &= 1 \\ \left\{x_r\right\}^{\mathrm{T}}\left[K\right]\left\{x_r\right\} &= \lambda_r\end{aligned}\right\}, \quad r = 1, 2, \ldots, n \tag{2.15}$$

Based on the definitions given by Eqs. (2.7) and (2.8), the orthogonality relations (2.12) and (2.13), and the normalization Eqs. (2.15) can be grouped in the following matrix equations, in terms of the spectral and modal matrices:

$$\left[X\right]^{\mathrm{T}}\left[M\right]\left[X\right] = \left[N\right] \tag{2.16}$$

$$[X]^{\mathrm{T}}[K][X]=[N][\Lambda] \tag{2.17}$$

where:

$$[N]=\mathrm{diag}\{\eta_1,\eta_2,\ldots,\eta_r\} \tag{2.18}$$

is the matrix of generalized masses.

2.3. Expansion Theorem

The orthogonality property of the eigenvectors demonstrated in the previous section ensure that, under the assumption of distinct eigenvalues, the eigenvectors of a n d.o.f. vibratory system form a set of n linearly independent vectors. In the context of Linear Algebra, this set is known to constitute a vector basis of the n-dimensional space which comprises the vectors that represent all the possible forms of motion of the system satisfying the prescribed boundary conditions. This means that any response of the system (free or forced motions) can be expressed uniquely as a linear combination of the n eigenvectors, as follows:

$$\{x(t)\}=\sum_{r=1}^{n}\{x_r\}c_r(t)=[X]\{c(t)\} \tag{2.19}$$

where $c_r(t)$ are the coefficients of linear combination, which are grouped in the vector $\{c(t)\}=\begin{bmatrix}c_1(t) & c_1(t) & \cdots & c_n(t)\end{bmatrix}^{\mathrm{T}}$.

Equation (2.19) expresses the *Expansion Theorem* or *Principle of Modal Superposition*, which represents the basis of all *Modal Analysis* procedures for linear mechanical systems.

2.4. Free Responses of Undamped Systems to Initial Conditions

For an n d.o.f. undamped system subjected to a set of prescribed initial conditions, the equations of motion are obtained from Eq. (2.1), resulting:

$$[M]\{\ddot{x}(t)\}+[K]\{x(t)\}=0 \tag{2.20}$$

satisfying:

$$\{x(0)\}=\{x_0\},\ \{\dot{x}(0)\}=\{\dot{x}_0\} \tag{2.21}$$

By using the Expansion Theorem, the solution of Eq. (2.20) is written as:

$$\{x(t)\} = [X]\{c(t)\} = \sum_{r=1}^{n} \{x_r\} c_r(t) \tag{2.22}$$

By introducing Eq. (2.22) into Eq. (2.20) and pre-multiplying the resulting equation by $[X]^{\mathrm{T}}$, one has:

$$[X]^{\mathrm{T}}[M][X]\{\ddot{c}(t)\} + [X]^{\mathrm{T}}[K][X]\{c(t)\} = 0 \tag{2.23}$$

Taking into account the orthogonality/norm relations (2.16) and (2.17), Eq. (2.23) becomes:

$$[N]\{\ddot{c}(t)\} + [N][\Lambda]\{c(t)\} = 0 \tag{2.24}$$

As $[N]$ and $[\Lambda]$ are diagonal matrices, Eq. (2.24) is comprised by n *uncoupled* second order differential equations of the type:

$$\ddot{c}_r(t) + \omega_r^2(t) = 0, \quad r = 1, 2, \ldots, n \tag{2.25}$$

Each equation in (2.25) is similar to the equation of motion of a single d.o.f. undamped system, the solution of which is given below:

$$c_r(t) = C_r \cos(\omega_r t) + D_r \sin(\omega_r t) \tag{2.26}$$

Then, by introducing Eq. (2.26) into Eq. (2.22), one writes:

$$\{x(t)\} = \sum_{r=1}^{n} \left[C_r \cos(\omega_r t) + D_r \sin(\omega_r t)\right]\{x_r\} \tag{2.27}$$

The $2n$ constants C_r and $D_r, r = 1,\ldots,n$ appearing in Eq. (2.27) are obtained by imposing the initial conditions, as follows:

$$\{x(0)\} = \{x_0\} = \sum_{r=1}^{n} C_r \{x_r\} \tag{2.28}$$

By pre-multiplying Eq. (2.28) by $\{X_s\}^{\mathrm{T}}[M]$, accounting for the orthogonality relations, one has:

$$\{x_s\}^{\mathrm{T}}[M]\{x_0\} = \sum_{r=1}^{n} C_s \{x_s\}^{\mathrm{T}}[M]\{x_r\} = C_s \eta_s \tag{2.29}$$

Then:

$$C_s = \frac{1}{\eta_s}\{x_s\}^{\mathrm{T}}[M]\{x_0\}, \quad s = 1,\ldots,n \tag{2.30}$$

By deriving Eq. (2.27) with respect to time, one obtains:

$$\{\dot{x}(t)\} = \sum_{r=1}^{n}\omega_r\left[-C_r\sin(\omega_r t) + D_r\cos(\omega_r t)\right]\{x_r\} \tag{2.31}$$

from which:

$$\{\dot{x}_0\} = \sum_{r=1}^{n}\omega_r D_r\{x_r\} \tag{2.32}$$

By following a procedure similar to that presented above for dealing with initial displacements, one has:

$$\{x_s\}^{\mathrm{T}}[M]\{\dot{x}_0\} = \sum_{r=1}^{n}\omega_r D_r\{x_s\}^{\mathrm{T}}[M]\{x_r\} = \omega_r D_s\eta_s \tag{2.33}$$

from which:

$$D_s = \frac{1}{\eta_s\omega_s}\{x_s\}^{\mathrm{T}}[M]\{\dot{x}_0\},\ s=1,2,\ldots,n \tag{2.34}$$

By associating Eqs. (2.27) and (2.30) and (2.34) one obtains a general expression for the system's response to an arbitrary set of initial conditions. It is worth mentioning that Eq. (2.27) states that the system's time response is a linear combination of the eigenvectors, in which the coefficients of linear combination are harmonic functions whose frequencies correspond to the natural frequencies of the system $\omega_r, r = 1,\ldots n$.

2.4.1. Systems with Rigid Body Modes

As discussed before, in the cases in which the kinematic constraints do not preclude the existence of rigid-body modes, the systems present a number p of null eigenvalues $(p \le 6)$. In such cases, for convenience, matrices $[X]$, $[\Lambda]$ and $[N]$ are partitioned according to:

$$[X] = \left[[X_1]_{p\times n} \qquad [X_2]_{n-p\times n}\right]$$

$$[N]=\begin{bmatrix}[N_1]_{p\times p} & 0\\ \\ 0 & [N_2]_{n-p\times n-p}\end{bmatrix}$$

$$[\Lambda]=\begin{bmatrix}[\Lambda_1]_{p\times p} & 0\\ \\ 0 & [\Lambda_2]_{n-p\times n-p}\end{bmatrix}$$

where the index 1 holds for the rigid body modes and the index 2 refers to the elastic modes.

By using the orthogonality relations (2.16) and (2.17), one writes:

$$[X]^{\mathrm{T}}[M][X]=[N] \;\rightarrow\; \begin{cases}[X_1]^{\mathrm{T}}[M][X_1]=[N_1]\\ [X_1]^{\mathrm{T}}[M][X_2]=[0]\\ [X_2]^{\mathrm{T}}[M][X_2]=[N_2]\end{cases} \tag{2.35}$$

$$[X]^{\mathrm{T}}[K][X]=[N][\Lambda] \;\rightarrow\; \begin{cases}[X_1]^{\mathrm{T}}[K][X_1]=[0] \;\Rightarrow\; [K][X_1]=0\\ [X_1]^{\mathrm{T}}[K][X_2]=[0]\\ [X_2]^{\mathrm{T}}[K][X_2]=[N_2][\Lambda_2]\end{cases} \tag{2.36}$$

For each rigid-body mode $\lambda_r=\omega_r^2=0,\; r=1,2,...,p$ corresponding modal equation of motion (2.25) simplifies to:

$$\ddot{c}_r(t)=0, \quad r=1,\ldots,p \tag{2.37}$$

whose solution is expressed as follows:

$$c_r(t)=C_r+D_r(t), \quad r=1,\ldots,p \tag{2.38}$$

As seen before, for the elastic modes, the solution is given by:

$$c_r(t)=C_r\cos(\omega_r t)+D_r\sin(\omega_r t),\; r=p+1,p+2,...n \tag{2.39}$$

Returning to Eq. (2.27), the complete solution is obtained:

$$\{x(t)\} = \sum_{r=1}^{p} (C_r + D_r t)\{x_r\} + \sum_{r=p+1}^{n} \left[C_r \cos(\omega_r t) + D_r \sin(\omega_r t) \right] \{x_r\} \tag{2.40}$$

By introducing the initial conditions, and following the procedure described previously in Section 2.5, one obtains the following expressions for the constants C_r and D_r :

$$C_r = \frac{1}{\eta_r} \{x_r\}^{\mathrm{T}} [M] \{x_0\}, \qquad r = 1, \ldots, n \tag{2.41}$$

$$D_r = \begin{cases} \dfrac{1}{\eta_r} \{x_r\}^{\mathrm{T}} [M] \{\dot{x}_0\}, & r = 1, \ldots, p \\ \dfrac{1}{\eta_r w_r} \{x_r\}^{\mathrm{T}} [M] \{\dot{x}_0\}, & r = p+1, \ldots, n \end{cases} \tag{2.42}$$

The association of (2.41), (2.42) and (2.27) provides the complete expression for the response to initial conditions of an n d.o.f system containing p rigid-body modes.

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Biographical Sketches

Valder Steffen, Jr is a Mechanical Engineer (UNICAMP, Brazil, 1976) and Doctor in Mechanical Engineering (University of Franche-Comté, France, 1979). He did his *Habilitation* at the University of Franche-Comté in 1991. He was a visiting scientist at the INSA de Lyon – France, in 1986-87 and a Fulbright Scholar at Virginia Tech – USA, in 1999-2000. He has advised 21 Master dissertations and 14 Doctorate theses. He is now Professor at the School of Mechanical Engineering, Federal University of Uberlândia, in Brazil, where he has been teaching System Dynamics, Mechanics, Rotor Dynamics, and Optimization Techniques to undergraduate and graduate students. His research interest is focused on dynamics of mechanical systems, optimization and inverse problems, and smart structures.

Domingos Alves Rade holds a Mechanical Engineer Degree (Federal University of Uberlândia, Brazil, 1984), a Master Degree in Aeronautical Engineering (Aeronautics Technology Institute, São José dos Campos, Brazil, 1987) and a Doctorate Degree in Sciences for the Engineer (University of Franche-Comté, France, 1994). He was a visiting professor at the University of Franche-Comté (2004) and at the National Institute of Applied Sciences of Rouen (2007). He has advised 10 Master dissertations and 8 Doctorate theses. He is currently an Associate Professor at the School of Mechanical Engineering of the Federal University of Uberlândia, Brazil, where he has been teaching Engineering Dynamics, Finite Elements, Mechanical Vibrations, Modal Analysis to undergraduate and graduate students. His current research interests are passive and active vibration control, structural health monitoring, smart material systems, fluid-structure interaction and stochastic modeling in structural dynamics.