SIMILITUDE AND THEORY OF MODELS

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Contents

Summary

The concepts of Similitude, Dimensional Analysis and Theory of Models are presented and used in this chapter. They constitute important theoretical tools that allow scientists from many different areas to go further on their studies prior to actual experiments or using small scale models. The applications discussed herein are focused on thermal sciences (Heat Transfer and Fluid Mechanics). Using a formal approach based on Buckingham's π-theorem, the paper offers an overview of the use of Dimensional Analysis to help plan experiments and consolidate data. Furthermore, it discusses dimensionless numbers and the Theory of Models, and presents a brief introduction to Scaling Laws.

1. Introduction

Generally speaking, similitude is recognized through some sort of comparison: observing some relationship (called similarity) among persons (for instance, relatives), things (for instance, large commercial jets and small executive ones) or the physical phenomena we are interested. The relation between small size structures (called models) installed inside wind tunnels or laboratories and human size structures such as buildings and airplanes (called prototypes) are often studied using the concept of Dimensional Analysis. This is an important theoretical tool that allows scientists to go much further in their studies as well as it enables them to predict world size scale phenomena such as the behavior of ships facing ocean thunderstorms or tsunamis using results obtained in laboratory pools in which waves are artificially generated. This article intends to describe how such extraordinary facts are achievable.

When these concepts started to be developed is not clear, but it is now recognized that the oldest known work bearing them was done by François Daviet de Foncenex (1734-1799), a disciple of Lagrange. Many distinguished scientists such as Newton, Fourier, Lord Rayleigh, P.W. Bridgman and Einstein used and recognized the importance of similarity and dimensional analysis. Much more recent and certainly worth mentioning are the studies made by the great British scientist G.I Taylor on the mechanical effects of a nuclear explosion - an example of the power of these concepts to disclosure the hidden physics inside complex phenomena. In such studies, G.I. Taylor replaced the real problem with an ideal one, which allowed him to overcome the lack of detailed information on many aspects of the problem (e.g. initial conditions). Doing so, he was able to obtain a remarkably good description of the time variation of the radius of the explosion generated shock wave in terms of the energy released at the explosion and other parameters. It constitutes an impressive example of concepts associated to Similitude, Theory of Models, Dimensional Analysis and scaling laws.

The use of Dimensional Analysis and Theory of Models are presented in this paper. Due to space limitations and considering the intended audience, fundamental topics such as historic developments, dynamic similarity and scaling laws are only briefly discussed. Many other aspects and examples are kept outside this text. Interested readers are invited to refer to the Bibliography for further details and applications from many different areas. The applications discussed here are focused on thermal sciences (Heat Transfer and Fluid Mechanics) due to the author’s experience.

2. Dimensional Analysis

Quite often, one finds himself having an adequate understanding of the physics involved in a problem under investigation. However, mathematical difficulties may prevent any reasonable chance of finding a closed-form solution and only extensive tabular data can be obtained through numerical methods. In such situations, Dimensional Analysis (or methods) may be used to help the investigation. Clearly, Dimensional Analysis is a theoretical tool that helps understanding physics as it allows the identification of the governing equations from the analysis of the dimensionality of the variables involved. The various procedures for establishing the dimensionless numbers or groups are all mathematically sound. Birkhoff, for instance, proved that it is part of the general theory of invariant parameters of equations.

Although there are other options, the formulation to be discussed, the Buckingham π-theorem, is one of the most commonly used. Herein, the approach proposed by Arpaci & Larsen will be briefly presented. According to these authors, this powerful method consists of two steps. In the first one, all dependent and independent variables must be made dimensionless in terms of inherent characteristic properties or some arbitrarily selected reference quantities. In the second step, all arbitrarily selected reference quantities are eliminated using the mathematical principle which states the invariance of the number of variables of a mathematical expression under any transformation, and the physical principle which states the dimensional homogeneity of a physical expression: Newton’s law must hold no matter the system of units, as nicely put by Stahl, pointing out the concept first noted by Fourier, in 1822.

2.1. Application

The application of the foregoing principles will be demonstrated in terms of the natural convection heat transfer from an isothermal sphere with external diameter D. As mentioned, the first step in Dimensional Analysis is to establish all the variables necessary to describe the phenomenon, such as the surface temperature (θ_s) and the ambient temperature (θ_∞). As the governing equation is supposed unknown at this point, some oriented guess is necessary to list all relevant variables. Once these are established, the dimensionless numbers may be determined. For the problem under consideration, the physics indicates that the variables are related to each other as:

$$h = f(D, g\beta\Delta\theta, \rho, \mu, c_P, k)$$

In the above equation, h is the convective heat transfer coefficient, g is the acceleration of gravity, β is the coefficient of thermal expansion, $\Delta\theta = \theta_s - \theta_\infty$ is the temperature difference, $g\beta\Delta\theta$ is the buoyancy term, ρ is the fluid density, μ is the fluid absolute viscosity, c_P is the fluid specific heat, and k is the fluid thermal conductivity. As the functional relation $f(...)$ is an unknown function, it will be kept that way throughout the procedure. Without planning, the investigation would involve studying how 6 variables affect the heat transfer coefficient, which is most certainly an impressive and time consuming task. Dimensional Analysis may help with this, as it will be shortly seen.

In terms of the fundamental units of mechanics mass [M], length [L], time [T] and temperature [θ], the above expression may be rewritten as:

$$h\left[\frac{M}{T^3\theta}\right] = f_1\left(D[L], g\beta\Delta\theta\left[\frac{L}{T^2}\right], \rho\left[\frac{M}{L^3}\right], \mu\left[\frac{M}{LT}\right], c_P\left[\frac{L^2}{T^2\theta}\right], k\left[\frac{ML}{T^3\theta}\right]\right)$$

The elimination procedure may be started several ways. For instance, let us rearrange the above equation to obtain a simpler one independent of one fundamental unit, say [M]. Picking ρ, for example, we combine (that is, multiply or divide) it with h, μ and k in such a way that the resulting combinations become independent of mass. Thus:

$$\frac{h}{\rho}\left[\frac{L^3}{T^3\theta}\right] = f_2\left(D[L], g\beta\Delta\theta\left[\frac{L}{T^2}\right], \frac{\mu}{\rho}\left[\frac{L^2}{T}\right], c_P\left[\frac{L^2}{T^2\theta}\right], \frac{k}{\rho}\left[\frac{L^4}{T^3\theta}\right]\right)$$

Next, let us eliminate [θ] picking $\left[k/\rho\right]$. This will result in:

$$\frac{h}{k}\left[\frac{1}{L}\right] = f_3\left(D[L], g\beta\Delta\theta\left[\frac{L}{T^2}\right], \frac{\mu}{\rho}\left[\frac{L^2}{T}\right], \frac{\rho c_P}{k}\left[\frac{T}{L^2}\right]\right)$$

By inspection, the simplest way to eliminate [L] is by D:

$$\frac{hD}{k} = f_5\left(\frac{g\beta\Delta\theta}{D}\left[\frac{1}{T^2}\right], \frac{\mu}{\rho D^2}\left[\frac{1}{T}\right], \frac{\rho c_P D^2}{k}[T]\right)$$

Finally, let us eliminate [T] by the group $\frac{\rho c_P D^2}{k}$, resulting in:

$$\frac{hD}{k} = f_6\left(\frac{g\beta\Delta\theta}{D}\times\left(\frac{\rho c_P D}{k}\right)^2, \frac{\mu}{\rho D^2}\times\frac{\rho c_P D^2}{k}\right)$$

$$= f_6\left(\frac{\rho c_P g\beta\Delta\theta D^3}{k}, \frac{c_P\mu}{k}\right)$$

Recalling that $\frac{k}{\rho c_P} = \alpha$ is the fluid thermal diffusivity, the above equation may be rewritten as:

$$\frac{hD}{k} = f_6\left(\frac{\rho c_P g\beta\Delta\theta D^3}{k}, \frac{c_P\mu}{k}\right) = f_6\left(\frac{g\beta\Delta\theta D^3}{\nu\alpha}, \frac{\nu}{\alpha}\right)$$

As it is known, the group $\frac{hD}{k}$ is the Nusselt number Nu (ratio between convection and conduction heat transfer). On the right hand side of the above equation two terms appear. The first one, $\frac{g\beta\Delta T D^3}{\nu\alpha}$, is known as the Rayleigh number, Ra, that is a ratio between buoyancy, viscous forces, enthalpy flow and conduction heat transfer $\left(\frac{buoyancy\ force}{viscous\ force}\times\frac{enthalpy\ flow}{conduction}\right)$. The second one, $\frac{\nu}{\alpha}$, is the Prandtl number, Pr, that is a ratio between viscous forces, enthalpy flow, inertia forces and conduction heat transfer $\left(\frac{viscous\ force}{inertial\ force}\times\frac{enthalpy\ flow}{conduction}\right)$. Therefore, the general yet still unknown law is:

$$Nu = f(Ra, Pr)$$

This expression will conduct the experiment as well as the display of any experimental results eventually obtained. Definitely, it is much simpler to conduct an experimental investigation using the above equation as it has only two variables, Ra and Pr. Sonin discusses the consequences of using an incomplete set of independent quantities and its fatal result and the less critical situation of the use of superfluous quantities.

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Biographical Sketch

Washington Braga holds a Bachelor degree in Mechanical Engineering (1975) and a Master degree (1978) in Thermo-Sciences from the Pontifical Catholic University of Rio de Janeiro, and a PhD in Mechanical Engineering (1985) from The University of Michigan, Ann Arbor. His current interests relates to engineering optimization and inverse problems associated with Thermal Sciences. Another area of permanent interest is Engineering Education.

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