CONTINUUM MECHANICS AS A GROUND FOR RHEOLOGY

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Contents

Summary

The basic principles of continuum mechanics have been introduced and their physical meaning has been discussed. The difference between physical and mathematical points is explained. The transition to infinitesimal values opens the way to treat a medium as a

continuum. Fundamental conceptions of stress, deformation and rate of deformation are considered as objects of tensorial nature. It allowed introducing invariants and principle values of the objects under discussion. The way of splitting any of these tensors into spherical and deviatoric components was shown. This is necessary for constructing general relationships between stresses and deformations. Main types of stress states and deformations – simple and pure shear, uniaxial elongation, hydrostatic pressure and compression – are discussed in details. The difference between deformation and displacement and between rate of deformation and vorticity is demonstrated. The special situation of large deformations was treated as the important case in rheology of real materials and flow of elastic liquids. It was shown that solving theoretical and applied problems in continuum mechanics is based on combined analysis of the stress balance equation and the laws of mass and energy conservation. This system of determining equations becomes closed in adding a rheological equation of state – constitutive equation. The latter is the relationships between stresses and deformations (rates of deformations). There are definite physical principles limiting freedom in the formulation of rheological equations of state.

1. Introduction

Classical continuum mechanics is one of the milestones of rheology. Indeed, rheology, dealing with *properties of a matter*, regards these properties as relationships between *stresses* and *deformations*, which are the fundamental concepts of continuum mechanics.

The idea of continuum, as well as mathematical operations used in mechanics, suppose that there is a continuous transition and movement from point to point. In this case a "*point*" is understood as a mathematical object of the infinitesimally small size. However, it is necessary to accept the following contradiction: a "physical" point is something different than a mathematical point.

Almost everybody is convinced that matter consists of molecules and intermolecular empty space that means that in reality, any material body is *heterogeneous*. At the same time, any observer is sure that he "sees" (if not to apply special optical methods) a body of matter as a *continuous* mass without holes and empty spaces. The obvious way out of these contradictory evidences lies in the idea of the *geometrical scale of observation*.

This scale must be large enough not to distinguish individual molecules or their parts. The characteristic order of sizes of a molecule (its cross-section or the length of several bonds) is 1 nm. Then, only when dealing with the sizes of the order of at least 10 nm, one can neglect molecular structure and treat a body as *continuous* and *homogeneous*. It means that a characteristic volume should be of an order larger than 10^3 nm^3. This is the real size of a physical "point", which is quite different from a philosophical or geometrical point. The latter is an infinitely small object of the zero-size. The physical "point" contains 10^4 molecules or segments of macromolecule, and throughout its volume all molecular-size fluctuations are averaged. The number of molecules in such a point is large enough for smoothing and averaging procedures.

In many cases, especially when discussing properties of a single-component material, it is possible to neglect the inherent structure of a medium and the difference between the ideas of the “physical” and “mathematical” concepts of a point is immaterial.

Having in mind the real scale of a physical point, it is supposed that there is a right to apply methods of mathematical analysis of infinitesimal quantities (which formally relate to a geometrical point) to a physical medium. The formal extrapolation of physics-based analysis to infinitely small sizes tacitly avoids the incorrectness of this operation, and the only justification for this is the fact that in almost all practical applications, nobody is interested in what really happens in a *very* small volume.

However, there are at least two important principal exceptions.

First, in some applications we meet with real "zero” size, if geometrical shapes have sharp angles then the size at the corner of any angle is (formally) equal to zero. The extrapolation of the calculation results to such “zero” volumes sometimes leads to infinite values, and this is out of the realm of physical meaning.

Second, there are many materials, which cannot be considered as homogeneous in principle, and it is necessary to take into consideration their *structure*, i.e. such materials are *heterogeneous* by definition. A very typical example of such heterogeneity is different suspensions, and filled polymers and materials with well-arranged (in reinforced plastics and cement) structure. In some applications the structure of heterogeneous materials may be out of interest and it is possible to continue to treat the medium as homogeneous, averaging inner differences in relation to much higher geometrical scale. For example, for many astronomic observations, Sun and our Earth are regarded as quite homogeneous and moreover can be treated as "points". In other cases, the role of heterogeneity can be important and even become a determining factor (for example, for reinforced plastics), but, in any case, the scale of such heterogeneity has to be much larger than the characteristic molecular sizes.

And finally, there is a special branch of rheology called *microrheology*, not directly based on continuum mechanics. The subject of microrheology is the study of the mechanical behavior of micro-objects, such as long-chain macromolecules, nano-particles and so on. The main instrument for examining these objects is statistical physics (as started by the classical works of Einstein and Smoluchowski, 1905-1911). These investigations try to explain macrorheological effects basing on conceptions of molecular movements and their goal is to establish the correspondence between micro- and macro-scale phenomena.

2. Stresses

Any external action applied to a body leads either to a movement of a body as a whole or to the distortion of its initial shape, or perhaps to both results simultaneously. The first consequence of an external action - the movement of a body in space and/or its rotation around its center of gravity, without any changes of its shape - is a subject of the study by mechanics, and as such is out of interest for rheology. Only what happens *inside* a body is of principal interest of our discussion

The applied *forces* regardless of the origin of these forces create the *dynamic situation* in any point of a medium, which is characterized by a physical object called *stress*. In some cases stresses can exist inside a body in the absence of external forces. These are, for example, the cases of thermal stresses appearing due to temperature inhomogeneity throughout a body or frozen stresses stored as a result of thermal and mechanical pre-history of a body and due to its heterogeneity.

2.1. General Theory - Introduction

An initial idea of stress is very simple and obvious. Let us consider a body exemplified by a long bar (Figure 1). The area of its normal cross-section is S. Let some extensional force be applied to the free end of a bar. If to cut a bar at the section S, the action of the one part to the other can be substituted by force F (or presented as a force). So, the conception of stresses is the *method* used in continuum mechanics *to describe the interaction* between different parts of a body.

The force, F, in the simplest case shown in Figure 1 acts normally to the surface, S. The result of the action of a force depends on the size of a body (or, in our case, on its cross-section). Therefore we need to introduce the specific force at any point of the section, which is equal to F/S and it is a *normal stress* or *tensile stress,* σ_{E}:

$$\sigma_{\mathrm{E}} = \frac{F}{S} \qquad (1)$$

i.e. *stress is a force per the unit of the surface area.*

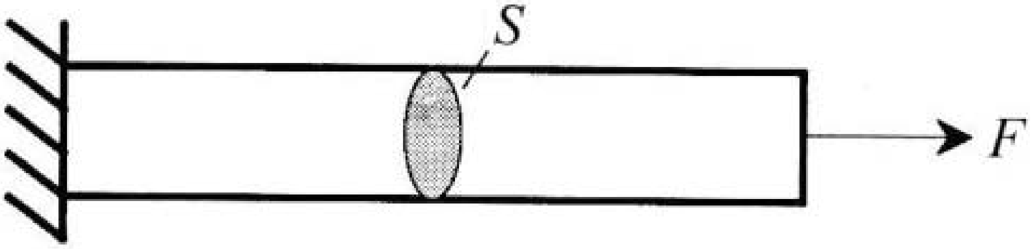

Figure 1. A bar loaded by a normal force.

The force at any surface may not be constant but distributed on this surface in an arbitrary manner, i.e., be a function of the coordinates. For example, a train moving along the rails presses on them at several local zones (where wheels touch a rail), i.e., a force is distributed along a rail and as a result a rather complex pattern of stresses acts in a rail.

In this case, as in many others, in order to avoid the force distribution in space a small area, ΔS, is marked. A relative (specific) force, ΔF, acts on the area, and $\Delta F/\Delta S$, is calculated. If to decrease the surface area, one eventually comes to the limit of this ratio:

$$\sigma = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S}, \quad \text{i.e. } \sigma = \frac{dF}{dS} \qquad (2)$$

This is a more general and a more exact definition of stress than Eq. (1). However, it is not the complete definition. Indeed, force at the area ΔS can be *oriented* along different directions, because force, in fact, is the *vector* value $\mathbf{F}$. This vector, as any other, can be decomposed into three components along three coordinate mutually perpendicular axes (in this book, only Cartesian coordinate systems will be considered); in particular it can be composed of one perpendicular and two tangential components to the area ΔS. The first component is the *normal stress*, and tangential components are the *shear stresses*.

Besides, the *orientation* of the small area ΔS can be also quite arbitrary inside a matter. This idea is expressed by a vector $\mathbf{n}$, which determines the orientation of ΔS being normal to it. So, the stress is the result of the combination of two vectors, $\mathbf{F}$ and $\mathbf{n}$, defined at any reference point. Then, the stress should be defined as the derivative $d\mathbf{F}/d\mathbf{n}$, and this illustrates the idea of the independence of the stress as the physical object on the choice of coordinate axis, because any vector is the physical object existing regardless of the choice of a coordinate system.

However, for practical applications it is convenient to operate not with a vector by itself but with its projections on coordinate axes. Any vector can be decomposed into its three projections on orthogonal coordinate axes; let it be Cartesian coordinates, x_1, x_2 and x_3 .

So, the complete characterization of the stress as a physical object requires identification of two vectors: a force and a normal to the surface, to which this vector is applied. The physical objects determined in such a manner are called *tensors*, and that is why the *stress is a value of the tensor nature*. (Stress tensor is a particular case of a more general concept of tensors. This particular case is called tensor of the second rank.)

Let both vectors, $\mathbf{F}$ and $\mathbf{n}$, defined at any reference point, be represented by their three projections along the orthogonal coordinate axes:

$$\mathbf{n} = \mathbf{n}(n_1, n_2, n_3)$$

$$\mathbf{F} = \mathbf{F}(F_1, F_2, F_3)$$

Figure 2. illustrates this situation.

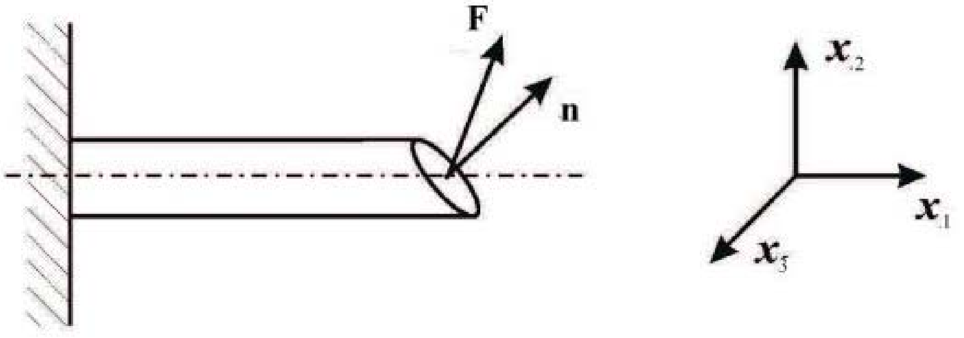

Figure 2. Vectors of a force $\mathbf{F}$ and an arbitrary orientation of the inclined plane $\mathbf{n}$.

Then the *nine* values can be defined: three projections of a force on surfaces determined by the three coordinate vectors. As the area of a surface is not essential, all values of $F_i (i = 1, 2, 3)$ must be divided by the area and it gives the components of a *stress tensor*, σ_{ij}, where the first index shows the orientation of a force and the second index designates the orientation of a surface.

The result is written in the form of a table (matrix)

$$\boldsymbol{\sigma} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} \qquad (3)$$

The meaning of the components of the stress tensor, σ_{ij}, is the following: the first column represents components of a force (of a vector of a force) applied at the plane normal to the x_1-axis, the second column gives the same for the x_2-axis and the third to x_3-axis. The projections of a normal to the surface are indicated by the second indices.

The table within the bars is the set (or a list) of all components (projections) of a force vector applied on differently oriented planes at an arbitrary point inside a body. In order to emphasize that this set of parameters presents a single physical object (physical reality) – the *stress tensor* it is usual to put them between the bars.

The picture showing all the components of the stress tensor acting at a point is drawn in Figure 3. One can easily see that components with the same numbers in the index are the normal stresses, quite equivalent to the initial definition of the normal stress in Eq. (1), and all values with different numbers in the index are the shear, or tangential, stresses.

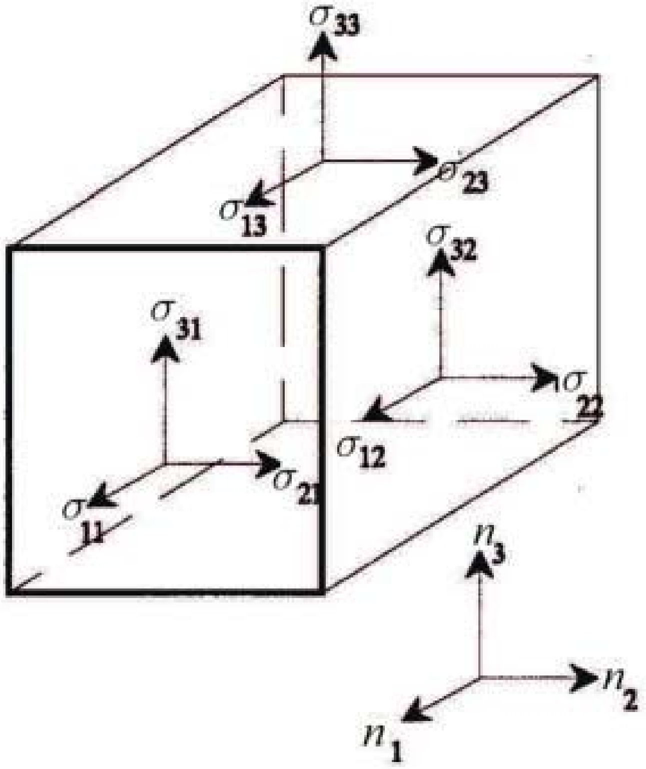

Figure 3. Three-dimensional stress state – definition of the stress tensor components.

The values of the stress tensor components depend on the orientation of the coordinate axes. Then the values in the tensor matrix can be quite different; in particular, they change with the rotation of the coordinate axes in space, though the stress state at this point, according to its physical meaning, remains the same. So, it is important to remember that, regardless of the choice of the coordinate axes, this is the same physical object, *invariant* to the choice of the coordinate axis.

2.2. Law of Equality of the Conjugated Stresses

Let us consider a plane section of a unit cube in Figure 3. The section is shown in Figure 4. Now only shear stresses are under discussion.

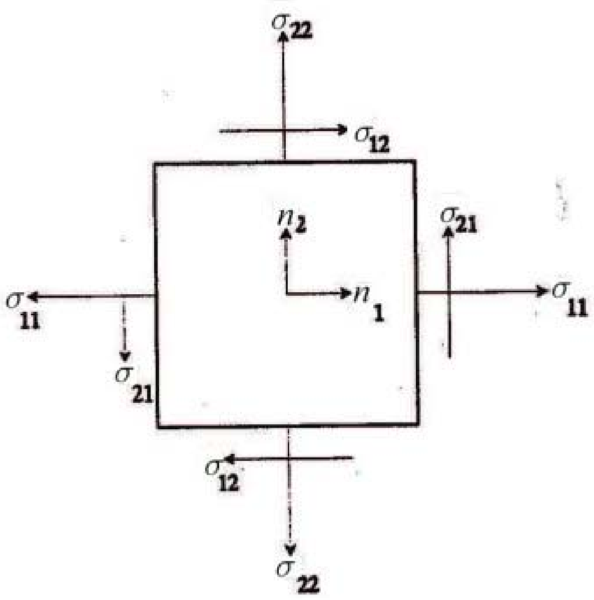

Figure 4. Two-dimensional (plane) stress state.

The rotational equilibrium condition about the central point of the square immediately gives the equality

$$\sigma_{12} = \sigma_{21}$$

The same is true for any other pair of the shear stresses. So, the general rule can be formulated as

$$\sigma_{ij} = \sigma_{ji} \qquad (4)$$

These equalities are known as the *Cauchy rule* of equality of the conjugated shear stress pairs (publication of 1827).

The result means that only three independent shear components of the stress tensor exist, and the stress state at a point is completely defined not by nine independent but by *six* values: three normal, $\sigma_{11}, \sigma_{22}, \sigma_{33}$, and three shear stresses, $\sigma_{12}, \sigma_{13}, \sigma_{23}$.

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Bibliography

Below, one will find the list of text-books and monographs, which give complete and advanced inside into the problem discussed in this chapter

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Biographical Sketch

Malkin Alexander Yakovlevich, born in Russia, January 31, 1937.

Graduated from Moscow (Russia) Technical University of Chemical Engineering as an engineer on polymer processing (1960).

Candidate of Sciences on Polymer Mechanics (1965); Doctor of Sciences on Polymer Physics (1971): Professor on Polymer Technology and Processing (1979) - The Highest Certifying Committee of the USSR.

He worked at the Institute of Chemical Engineering (1960-1962); Institute of Petrochemical Synthesis of the Russian Academy of Sciences (1962-1975, 2000- present), Research Institute for Plastics (1975-2000). Visiting Professor at Louisiana State University, USA (1992); Honorable Speaker for the Polymer Science by Chanchun Institute of Applied Chemistry, Chinese Academy of Science (2005), Consultant and Research Assistant at the Flow Process Research Center, Cape Peninsula University of Technology, Cape Town, Republic of South Africa (2002 – present).

He is the author of 5 monographs and more than 100 papers in English and 8 monographs and more than 300 papers in Russian, including:

- Rheology of Polymers (with G.V. Vinogradov), Springer, Heidelberg, Germany, 1980

- Rheokinetics (with S.G. Kulichikhin), Hüthig & Wept, Heidelberg, Germany, 1996

- Rheology. Conceptions, Methods, Applications (with A.Isayev), ChemTec Publ. Toronto, Canada, 2006

He took part as an invited plenary or keynote speaker at many International and National Conferences and Meetings on rheology, polymer processing and physics in Bulgaria, Canada, China, France, Germany, Great Britain, Israel, Poland, Portugal, Russia, South Africa, Spain, USA, and others

He was or is a member of Editorial Boards of several International Journals (Rheologica Acta, J. Non-Newtonian Fluid Mech., Intern. Polymer Processing, Intern J. Polymeric Materials, et al).

Field of studies – rheology of different substances, physics of polymers, polymer processing, polymer technology, mathematical modeling.

Prof. Dr. Sci., Malkin, Member of Russian Rheological Soc., American Soc. of rheology, Representative of the Russian Rheol. Soc. for International Committees on rheology.