BASIC LAWS AND PRINCIPLES OF QUANTUM ELECTROMAGNETISM

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Contents

Summary

The interaction of macroscopic objects with electromagnetic fields can be well-described using Faraday's laws and Maxwell's equations. In this approach, charges and

currents are subject to a smoothly varying force, and follow well-defined paths. These descriptions, however, are no longer adequate when applied to microscopic particles, such as electrons. The electron's behavior must be described by a quantum mechanical wavefunction, which only provides a probabilistic description of the electron's trajectory. Moreover, the interaction can no longer be seen as a continuous process. Instead, charged particles interact by emitting and absorbing discrete quanta of the electromagnetic field, known as photons. Special relativity allows the conversion between energy and mass, and when relativity is combined with quantum mechanics, creation and annihilation of particle-antiparticle pairs results. The Dirac equation, which is the result of constructing a wave equation for the electron which obeys special relativity, provides a natural explanation not only of the existence of antiparticles, but also of properties such as spin. In order to interpret the equation for a system where the number of particles is not constant, it is necessary to move away from classical wave mechanics to a field theory. Fields represent the particles such as electrons and photons, and operators acting on these fields return the numbers and properties (such as momentum and polarization) of the particles concerned. Gauge invariance requires that the physical properties of a system do not change when parameters of the underlying field theory (such as the phase) undergo arbitrary changes. Imposing the requirement of such invariance on a theory ensures that meaningful calculations can be performed which are not subject to infinite corrections. In the case of the Dirac equation, the requirement of local gauge invariance is identical to requiring that the particles undergo interactions with a structure which is exactly equivalent to that of electromagnetism.

1. Introduction

Quantum electromagnetism, or quantum electrodynamics as it is more often known, arises from the marriage of a quantum mechanical approach to the description of nature with the requirements of special relativity. Classically, charged particles follow well-defined trajectories, and interact with each other as a result of the Coulomb force between them, which can be parameterized as an electric field. In quantum mechanics, the defined trajectories are replaced by wave functions, which give the probability density for a particle being found at a particular place and time; the smooth potential is retained. However, there are a number of reasons for wishing to incorporate the principles of special relativity. First, all physical process should obey Lorentz invariance, and not be dependent on a particular frame of reference. Second, when low-mass particles such as electrons are studied, small amounts of energy can result in velocities comparable with the speed of light, where a classical expression for kinetic energy, for example, does not apply. Finally, even at modest speeds, charged particles experience the effects of both electrostatic and magnetic fields, and again a consistent description should be employed which is correct in all frames of reference. Quantum electrodynamics (QED) provides such a description, and results in the quantization of the field, in addition to the quantum mechanical portrayal of the particle’s behavior. (This is sometimes known as second quantization.) The quanta of the electromagnetic field are photons. Charged particles interact by the emission and absorption of one or several virtual photons, the momentum transferred by the photons corresponding to the action of a force between the particles.

In this chapter, after a brief description of the notation which is employed to describe relativistic quantities, the elements of a relativistic description of quantum electromagnetism are successively introduced. First, a relativistic expression is used for the relationship between energy and momentum, in place of that employed in the Schrödinger equation, and used to derive the Klein-Gordon equation for a free particle. Problems with the interpretation of this formalism are discussed, and an alternative, the Dirac equation, derived. It is shown that this requires a 4-component wave function, which can only be satisfied with the introduction of an intrinsic spin of $\frac{1}{2}\hbar$ and the existence of antiparticles. In a brief interlude, the properties of angular momentum and spin in quantum mechanics are outlined, and the Pauli matrices introduced. Dirac's association of holes in the vacuum sea with antiparticles will be contrasted with Feynman's more modern interpretation of antiparticles, and their representation in Feynman diagrams.

Next, interactions are introduced, through the use of the vector electromagnetic potential. The use of a Lagrangian approach to perform calculations in quantum mechanics is described, albeit rather briefly, and fields are introduced to allow for changing numbers of particles and quantized interactions. The principles of gauge invariance are briefly outlined, along with renormalizability, and it is shown that electromagnetic-like interactions are a natural consequence of demanding local gauge invariance. Finally the relationship between QED and other gauge interactions is summarized.

The last section of the chapter discusses experimental evidence for the validity of quantum electrodynamics, through the comparison of precision measurements and calculations of the magnetic moment of charged leptons such as electrons and muons.

2. Notation

Four-vectors are a convenient way of representing the various quantities which are related by Lorentz transformations. Examples of such quantities are time and space; energy and momentum; electrostatic and magnetic vector potential. The space-time 4-vector is thus written as

$$x^{\mu} \equiv (t, x, y, z) \equiv (t, \mathbf{r}). \tag{1}$$

Note the use of the Greek superscript, μ, to indicate that x^{μ} is a 4-vector. μ ranges from 0, representing the scalar part (here, t) to 3; components 1 to 3 represent the vector part $\mathbf{r}$). Note also that it is conventional to use units such that c, the speed of light, has the value 1, and so does not appear explicitly in the formula for x^{μ}.

The dot-product of two (ordinary) vectors is rotationally invariant. For example

$$\mathbf{a} \cdot \mathbf{b} \equiv a^i b^i. \tag{2}$$

Note here that a Roman superscript, i, is used to index the components of a 3-vector; i takes a value from 1 to 3. Note also the use of the Summation Convention – where a subscript is repeated, this implies the summation over all possible values. Explicitly, then,

$$\mathbf{a}\cdot\mathbf{b} \equiv a^i b^i \equiv \sum_{i=1,3} a^i b^i \equiv a^1 b^1 + a^2 b^2 + a^3 b^3 . \tag{2a}$$

The dot-product of two 4-vectors is similarly a Lorentz invariant. However, in this case, we must define the product of 4-vectors a^μ and b^μ as

$$a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \equiv a^\mu b_\mu \equiv a_\mu b^\mu \equiv a^\mu g_{\mu\nu} b^\nu , \tag{3}$$

where $a_\mu \equiv g_{\mu\nu} a^\nu \equiv a^\nu g_{\mu\nu}$, (4)

and $g_{\mu\nu}$ is the metric tensor:

$$g_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{5}$$

The summation convention in this case is that repeated *Greek* indices, one being superscript and the other subscript, are summed over the values 0 to 3 as indicated.

We must also introduce the differential operator

$$\partial_\mu \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \right) \equiv \left(\frac{\partial}{\partial t}, \nabla \right) . \tag{6}$$

For consistency with (4) above, we also have

$$\partial^\mu \equiv \left(\frac{\partial}{\partial t}, -\nabla \right) . \tag{6a}$$

Operators used in quantum mechanics do not always commute. That is, for two operators $\hat{A}$ and $\hat{B}$, the effect of $\hat{A}\hat{B}$ is not necessarily the same as $\hat{B}\hat{A}$. The commutator of $\hat{A}$ and $\hat{B}$ is defined as

$$\left[\hat{A}, \hat{B} \right] = \hat{A}\hat{B} - \hat{B}\hat{A} , \tag{7}$$

while the anticommutator is

$$\left\{ \hat{A}, \hat{B} \right\} = \hat{A}\hat{B} + \hat{B}\hat{A} . \tag{7a}$$

3. Relativistic Wave Equations

3.1. From Schrödinger to Klein-Gordon

The familiar low-energy wave equation is Schrödinger's equation

$$E\psi = i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right)\psi \tag{8}$$

where ψ is the wave function, E is the total energy, V the potential energy, m the mass of the particle concerned and $\hbar$ is Planck's constant h divided by 2π. When one considers a "generic" plane wave form for the wave function (with wave-vector $\mathbf{k}$ and angular frequency ω)

$$\psi = e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \equiv e^{i(\mathbf{p}\cdot\mathbf{r}-Et)/\hbar} \tag{9}$$

one can see that the first term on the right hand side of (8) corresponds to the classical expression for kinetic energy, $T = \dfrac{p^2}{2m}$, where p is the magnitude of the particle's momentum, $\mathbf{p}$. Schrödinger's equation is simply an expression for the total energy

$$E = T + V \tag{10}$$

where operator substitutions have been made for E and $\mathbf{p}$

$$\hat{E} \to i\hbar\frac{\partial}{\partial t} \qquad \hat{\mathbf{p}} \to -i\hbar\nabla\,. \tag{11}$$

(In subsequent expressions, the standard convention will be adopted of assuming that natural units are used where $\hbar$, like c, is equal to one and so does not appear explicitly in the formalism.)

Equation (8) is clearly not Lorentz invariant, containing derivatives which are first order in t but second order in $\mathbf{r}$. The correct relativistic relationship between energy and momentum is

$$E^2 = p^2 + m^2\,. \tag{12}$$

(For now, the potential energy term, corresponding to interactions, will be ignored. This will be re-introduced later.) Employing the operator substitutions of (11) in (12), we therefore have

$$\left(i\frac{\partial}{\partial t}\right)^2\varphi = \left(-i\nabla\right)^2\varphi + m^2\varphi \tag{13}$$

or $\left(\partial^{\mu}\partial_{\mu}+m^{2}\right)\varphi=0$. (13a)

This manifestly covariant expression is known as the Klein-Gordon equation. However, Dirac pointed out a problem with this formalism. It is second order in time, which means that φ cannot be interpreted as a wave function in the normal way. In particular, the probability density should be given by $i\left(\varphi^{*}\frac{\partial\varphi}{\partial t}-\frac{\partial\varphi^{*}}{\partial t}\varphi\right)$ rather than by $\varphi^{*}\varphi$, and such an expression is not guaranteed to be positive definite. Though with an appropriate interpretation of the wave function, the Klein-Gordon equation can be used to give a perfectly adequate description of spinless particles, such concerns led Dirac to search for an alternative relativistically invariant wave equation.

3.2. The Dirac Equation

Dirac suggested that it should be possible to write down a Lorentz invariant theory which is first-order in *t*. One could consider using the square root of (12),

$$p^{0}=\pm\sqrt{\mathbf{p}^{2}+m^{2}}\,. \tag{14}$$

However, it is clear that p^0 and $\mathbf{p}$ are not treated in an equivalent fashion. Also, taking the square root of an operator presents problems. Dirac therefore proposed a linear Hamiltonian (or energy operator)

$$\hat{H}_{\mathrm{D}}=i\frac{\partial}{\partial t}=\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}+\beta m\,. \tag{15}$$

Note that $\boldsymbol{\alpha}$ must be a vector, to preserve rotational invariance. Also, for a wave function in an eigenstate of energy, applying the Hamiltonian (15) must return a value compatible with the expression for energy in (12) above. Applying $\hat{H}_{\mathrm{D}}$ twice, we obtain

$$\left(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}+\boldsymbol{\beta}m\right)\left(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}+\boldsymbol{\beta}m\right)=E^{2}=\mathbf{p}^{2}+m^{2} \tag{16}$$

i.e. $$\begin{aligned}&\left(\alpha_{1}p^{1}+\alpha_{2}p^{2}+\alpha_{3}p^{3}+\beta m\right)\left(\alpha_{1}p^{1}+\alpha_{2}p^{2}+\alpha_{3}p^{3}+\beta m\right)\\&\qquad=\left(p^{1}\right)^{2}+\left(p^{2}\right)^{2}+\left(p^{3}\right)^{2}+m^{2}\end{aligned}\quad\ldots \tag{16a}$$

This must be true for *any* wave function, with any value of p^i, so we may equate coefficients:

of $\left(p^{1}\right)^{2},\left(p^{2}\right)^{2},\left(p^{3}\right)^{2},m^{2}\ \Rightarrow\ \alpha_{1}^{2}=1;\ \alpha_{2}^{2}=1;\ \alpha_{3}^{2}=1;\ \beta^{2}=1$ (17)

of $p^{1}p^{2}\left(\equiv p^{2}p^{1}\text{ since }p^{i}p^{j}\text{ commute}\right)\ \Rightarrow\ \alpha_{1}\alpha_{2}+\alpha_{2}\alpha_{1}=0$. (17a)

Similarly, other pairs of α_i, α_j must anticommute; $\left\{\alpha_i, \alpha_j\right\} = 0$ for $i \neq j$. Overall, this can be summarized as

$$\left\{\alpha_i, \alpha_j\right\} = 2\delta_{ij}\ ;\ \left\{\beta, \alpha_i\right\} = 0\ ;\ \beta^2 = 1 \tag{17b}$$

where we are employing the Kronecker delta symbol, which has a value of 1 if its two arguments are identical and 0 otherwise. Such relationships are clearly not possible if α_i and β are ordinary numbers.

In fact, Dirac showed that these relationships cannot be satisfied by 2×2 or 3×3 matrices either. The simplest possibility is 4×4 matrices. A possible set of such matrices is given by

$$\boldsymbol{\alpha}_i = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}_i \\ \boldsymbol{\sigma}_i & \mathbf{0} \end{pmatrix};\ \boldsymbol{\beta} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} \tag{18}$$

where the $\boldsymbol{\sigma}_i$ are the 2×2 Pauli matrices, which will be defined in the next section, $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{I}$ is the 2×2 unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Now the Dirac equation (15) can be written

$$i\frac{\partial \boldsymbol{\psi}}{\partial t} = \left(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \boldsymbol{\beta} m\right)\boldsymbol{\psi}\,. \tag{19}$$

Since $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}$ are 4×4 matrices, this implies that $\boldsymbol{\psi}$ must be a 4-component column vector. The significance of the 4 components of the wave function will be discussed in Section 5, after we have examined the representation of angular momentum in quantum mechanics.

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Biographical Sketch

Dr. Christopher Booth is a senior lecturer in High Energy Particle Physics at the Department of Physics and Astronomy of the University of Sheffield. He graduated from the University of Cambridge in 1976, before starting research for his PhD, also with the University of Cambridge. This was based on a study of the difference between antiproton-proton and proton-proton interactions, using the Hybrid Bubble Chamber Facility at the Stanford Linear Accelerator Center. Between 1979 and 1985, he was employed as a Research Associate with Cambridge, working at CERN (the European Laboratory for Particle Physics) on experiment UA5 at the proton-antiproton collider. His responsibilities were for the online trigger system and a small calorimeter used for detection and identification of neutral hadrons. Data analysis included studies of charged particle multiplicity distributions and correlations.

In 1985, Dr. Booth obtained a CERN Fellowship to work on experiment UA2. Here he was involved in the development of a new type of tracking detector using scintillating plastic fibers, read out through multi-stage gated image intensifiers and charge-coupled devices. He also implemented a distributed monitoring system using remote procedure calls to enable communication between microprocessors in the detector readout and the main data acquisition computer.

In 1988, Dr. Booth took up his present post, and started work on the ALEPH experiment at LEP. He is responsible for interface electronics between the accelerator and the experiment, and has undertaken analysis of multi-lepton and photonic final states, including tests of electron substructure and deviations from quantum electrodynamics. His present research involves searches for supersymmetry. He is also a member of the HARP collaboration, studying low energy hadron production in preparation for a future muon collider or neutrino factory. He teaches courses on Particle and Nuclear Physics and Particle Detectors. Dr Booth is married, with two daughters.