MODELING TRANSPORTATION NETWORKS VIA PRINCIPLES OF OPTIMALITY

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Keywords: spatial networks, transportation infrastructures, urban structure, roads, streets.

Contents

Summary

This chapter presents different models and a brief review of the literature about the applicability of notions of optimality in the modeling of the large-scale properties of transportation networks. The models discussed are to a large extent only idealizations of real transportation networks and focus on their general, large-scale properties. However, an attempt is be made to underline what specific characteristics of real transportation networks have inspired our modeling effort.

1. Introduction: Transportation Networks and General Properties of Spatial Networks

The main purpose of the present chapter is to present some of our studies on the applicability of notions of optimality in the modeling of the large-scale properties of transportation networks. The models we will discuss are to a large extent only idealizations of real transportation networks, and aimed at highlighting general properties rather than focusing on specific applications. We will nevertheless try to underline what specific characteristics of real transportation networks have inspired our modeling effort, and, in the case of road networks discussed below, we will debate more at length to what degree simplified models can capture features of real networks.

A secondary purpose of our presentation is to briefly review some literature about optimal transportation networks that could help to frame our work in a wider context. Given the breadth of the field of transportation network, our review is partial at best, and possibly biased towards those pieces of work that are more closely related to our approach.

Before proceeding it will be useful to briefly contextualize our approach in the broader field of network science.

During the 1990s we have witnessed to an explosion in the interest for complex networks, but the adoption of optimality principle in the study of such networks has been pretty limited. There are possibly good reasons behind the scarce popularity of approaches based on optimization. The notion of optimality immediately implies the existence of a generic “objective” function to be either minimized or maximized and a “mechanism” or “agent” that is wise enough to identify the objective function and powerful enough to drive the realization of the optimal network. It is sufficient to consider a paradigmatic example of complex network such the WWW, that has been shaped through the uncoordinated action of countless individuals, to understand how improbable would it be to claim the existence of an objective function, or that any kind of optimal topology may have been achieved without the intervention of some regulating agency. In other words, one may have the impression that principles of optimality can be usefully invoked only when the network under scrutiny is the result of “design.” Hopefully, the brief review and the work presented here may help showing that principles of optimality have a range of applicability wider that one could think at first sight and they are not incompatible with the notion of complexity, and do not necessarily imply the existence of a mighty supervising agent.

This chapter is organized as follows. In the next section we will briefly mention examples of transportation networks for whose study optimality principles have been successfully employed. In Section 3 we will deal with an abstract model of transportation network inspired by studies about the airport network and the physical Internet. We will assume that the network of interest realize the minimum of an objective function that depends quite generally from the topology of the network and on the fluxes that the network supports. In Section 4 we will discuss the topological transition between different classes of networks that occurs at varying the parameters that characterize the objective function. In Section 5 we will finally relax the assumption of global “optimum” and discuss a model for the formation of road networks based on a local (in time and space) optimization principle.

2. Optimal Transportation Network

Quite generally, a transportation network is a natural or artificial structure that conveys a specific quantity between geographically separated nodes. This definition is so broad that it covers an exceedingly large number of systems. Road networks, subways and train lines, rivers, electric circuits, circulatory systems in mammalians, pipelines, the Internet are just some of the example that could possibly be brought under such a wide umbrella. Principles of optimality have been invoked with explicative power in several of the examples mentioned above.

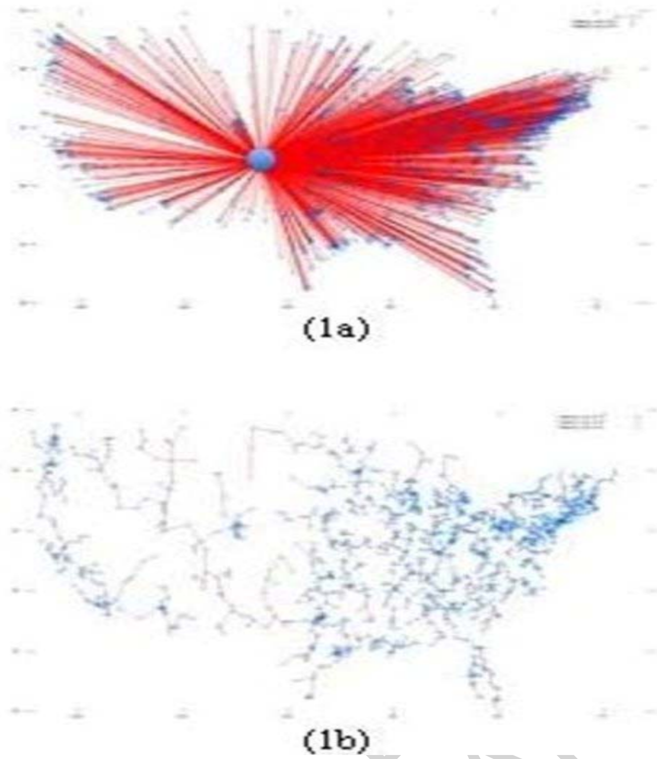

Figure 1. Two “unconstrained” example of network optimization with objective depending on Euclidean distance. In the left panel the average distance from a fixed point (blue) is minimized. Close-by points are connected to the reference point by independent roads, implying a huge total length. In the right panel the minimum spanning tree connecting the same set of points. A minimal total length implies a very long path between any two generic points.

A principle of optimality can be described via an objective (cost) function from a specified set of networks to the set of real numbers. From a design point of view, one’s task is to select the network that minimize such function and satisfies a (optional) set of constrains. The technical problem in finding the optimal network obviously depends on the complexity of the function to be minimized and on the set of constrains to be satisfied. In relevant applications the complexity of the cost function is usually a consequence of the attempt of compromising between antagonist requirements. This fact is illustrated in Fig. 1 with an example inspired from a 1972 book of P. S. Stevens.

In the left panel a set of points is to be connected in order to minimize the average (over all nodes) distance to a given reference point. If no other request are made the optimal solution is obviously realized by the star network centered in the reference point. Such solution optimizes the distance to the reference, but is clearly very expensive from the point of view of total length, since it provides independent connections also to points that are very close to each other. In the right panel is shown instead the shortest network that provides connectivity to all nodes. Such network realize an instance of Minimum Spanning Tree and it can be constructed by ranking the distances between all couple of points from shortest to longest, and, starting form the shortest link, progressively adding links with increasing length, while disregarding those that happen to close a loop. The minimum spanning tree is obviously convenient from the point of view of the total length, but the typical path between two generic nodes is extremely intertwined and long. Indeed Gastner and Newman have shown that efficient networks that simultaneously have a short total length and provide efficient connectivity can be found [Gastner]. Another classical example that involves minimizing distances is the problem of the Steiner tree, where — as for the MST— the shortest network that provides global connectivity is searched, but auxiliary nodes can be added. Steiner trees are characterized by the fact that three links separated by a 120 degrees angle are incident on a generic auxiliary node. To find the Steiner tree for a generic set of points is known to be an NP complete problem.

Both the problem of optimal traffic on a network and of optimal networks has a long tradition in mathematics and physics.

Problems of optimization for transportation networks often involve the optimization of the flow (or traffic) supported by the network. A classical example is that of determining currents in a resistor network. After Kirchhoff, this is achieved solving continuity equations for each node (the algebraic sum of currents entering a node is zero), together with the equations that state that the current in a link is proportional to the drop in electric potential at the ends of the link. It can be easily shown that this is equivalent to minimizing the sum (over all links) of the square the currents (with the continuity constrains). In this approach the electric potentials are easily recovered and found to play the role of Lagrange multipliers, and the objective function corresponds to the energy dissipated by the circuit.

More relevant to human mobility are the problems of traffic optimization on road networks. The problem has a long tradition and obvious importance in the field of civil engineering, possibly predating the seminal work by Wardrop in 1952. In these variational approaches, the cost per agent associated to traversing a link of the road network is usually a non-linear function of the total traffic supported by the link. In several circumstances the pattern of traffic that realize the Nash equilibrium (in this context a local minimum in which no single agent can change its route to a given destination without increasing its personal cost) is found to differ greatly from the global optimum. This circumstance is closely related to the well-known Braess paradox, the observation that removing specific links from a road network may lead to better traffic conditions. The paradox has found several empirical confirmations.

There are occasions in which the simultaneous optimization of the network and its flows are required, as in one of the models discussed in the next sections. An example of this class of models occurs in the study of river networks. It has been shown that the stationary solutions of a class of differential equations that describe the evolution of the system landscape/river network are realized by the so-called Optimal Channel Networks that minimize an objective function dependent on the fluxes and that is related to the dissipated energy by the flow of water. From the point of view of its usefulness, the variational approach allows a direct calculation of the exponent characterizing the power-law statistical distributions that describe the self-similar nature of river networks. More generally, Banavar and collaborators have shown that a wide class of physical phenomena related to transport can be re-interpreted with explicative power in a context of fluxes optimization.

In the examples discussed above, the rationale to invoke optimality principle is either the fact that optimality is a direct consequence of physical laws (not an infrequent case in physics), as in the case of resistor and river networks, or is a consequence of specific choices made by agents that try to minimize some generic cost, as in the of traffic on road networks. There is a third main reason that can possibly justify the adoption of variational approaches: evolution. In the biological realm, in fact, is not unconceivable that efficient transportation structures arise as a result of adaptation or of natural selection. Under this umbrella it is worth remembering the work on circulatory systems in mammalians by MacMahon, the explanation for the 4/3 power law in allometric scaling by Banavar *et al.* (note that alternative hypothesis have been proposed by West *et al.*), and the study of food webs by Garlaschelli *et al.* A last example worth mentioning is that of metabolic networks. Price *et al.* have shown that specific pathways maybe discovered if conditions for optimal growth are assumed.

With the exceptions of food webs and metabolic networks, all the studies mentioned above share the fact that the nodes of the network are embedded in a d-dimensional Euclidean space which implies that the degree is almost always limited and the connections restricted to `neighbors' only. A second broad class of optimal networks where spatial constraints are absent has been also investigated. It has been shown, for example, that optimization of both the average shortest path and the total length can lead to small-world networks, and more generally, degree correlations or scale-free features can emerge from an optimization process. Cancho and Sole' showed that the minimization of the average shortest path and the link density leads to a variety of networks including exponential-like graphs and scale-free networks. Guimera *et al.* studied networks with minimal search cost and found two classes of networks: star-like and homogeneous networks. Finally, Colizza *et al*. studied networks with the shortest route and the smallest congestion and showed that this interplay could lead to a variety of networks when the number of links per node is changed.

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Bibliography

Ahuja, R.K., Magnanti, T.L., and Orlin, J.B. (1993). *Network, Flows*, Prentice Hall, New Jersey [A complete introduction to problems related to flows on networks, both from the point of view of theory and algorithms.]

Banavar, J.R., Colaiori, F., Flammini, A., Maritan A., Giacometti, A., and Rinaldo, A. (1997). Sculpiting of a Fractal River Basin, *Physical Review Letters* 78, pp. 4522–4525 [Proves how river network configurations produced by erosion satisfy the independently formulated optimality principle at the basis of the Optimal Channel Network model]

Banavar, J.R., Colaiori, F., Flammini, A., Maritan, A., and Rinaldo, A. (2000) Topology of the Fittest Transportation Networks, *Physical Review Letters* 84, pp. 4745–4748 [Shows how different physical systems can be described through fluxes that satisfy a general principle of optimization.]

Banavar, J.R., Maritan, A., and Rinaldo, A. (1999). Size and form in efficient transportation networks, *Nature* 399, pp. 130–132 [It shows how the 4/3 scaling of metabolic rtes with organisms mass can be derived from a principle of optimization analogous to the one discussed in the ref. above.]

Barrat, A., Barthelemy, M., Pastor-Satorras, R. and Vespignani, A. (2004). The architecture of complex weighted networks, *Proceedings of the National Academy of Science*, 101, pp. 3747–3752. [A paper on the structure of weighted networks with a case study on the worldwide airport network.]

Barrat, A., Barthelemy, M. and Vespignani, A. (2005). The effects of spatial constraints in the evolution of weighted complex networks, *Journal of Statistical Mechanics*, P05003 [How space affects the properties of complex networks.]

Barthelemy, M., Flammini, A., Modeling urban street patterns. Phys. Rev. Lett. 100, 138702 (2008) [A model for street patterns based on local optimization.]

Berg, J. and Lassig M. (2002). Correlated Random Networks, *Physical Review Letters*, 89, 228701 [A model for networks with degree-degree correlations.]

Braess, D., Nagurney, A., and Wakolbinger, T. (2005). On a paradox of traffic planning, *Journal Transportation Science*, 39, pp. 446-450 [On the paradoxical fact that removing a road can improve the traffic.]

Cardillo, A., Scellato, S., Latora, V., Porta, S., Structural properties of planar graphs of urban street patterns. Phys. Rev. E 73, 066107 (2006). [This paper presents empirical results on urban street networks].

Colizza, V., Banavar, J.R., Maritan, A. and Rinaldo A. (2004). Network structures from selection principles, *Physical Review Letters*, 92, 198701 [Studies a class of network that simultaneously minimizes the path length between nodes and the total traffic congestion.]

Doyle, G. and Snell, J.L. (1989). *Random Walk and Electric Networks*, American Mathematical Society, Providence [A beautiful little book that establishes connections between problems related to random walks on networks and currents on resistor networks. It also shows how Kirchoff's law can be derived from the minimization of the dissipated energy.]

Ferrer i Cancho, R., and Solé, R.V. (2003). *Optimization in complex networks*, Statistical Mechanics of Complex Networks, *Lecture Notes in Physics*, 625, pp. 114–125, Springer, Berlin. [A review paper on different types of networks that can be obtained with an optimization principle.]

Graham, R. L. and Hell, P. (1985). On the History of the Minimum Spanning Tree Problem. *Annals of the History of Computing* 7, pp. 43–57 [Title is self-explicative].

Guimera, R., Diaz-Guilera, A., Vega-Redondo., F., Cabrales, A., and Arenas A. (2002). Optimal network topologies for local search with congestion, *Physical Review Letters*, 89, 258701 [Takes explicitly into account the issue of congestion in optimal traffic networks.]

Hwang, F.K., Richards, D.S., Winter P. (1992). *The Steiner Tree Problem*, Elsevier, North-Holland [About Steiner trees: the shortest wiring diagram that connect physically separated nodes.]

Jungnickel, D. (2007). *Graphs, Networks and Algorithms*, Springer-Verlag, New York [A pretty self-contained textbook in combinatorial optimization and its relation to graph theory.]

Kolata, G. (1990). What if They Closed 42nd Street and Nobody Noticed? New York Times, Dec. 25th [Reports about a real example of Braess paradox.]

Lammer, S., B. Gehlsen, D. Helbing, Scaling laws in the spatial structure of urban road networks. *Physica A* 363, 89, (2006). [An empirical study of street networks in German cities].

Maritan, A., Colaiori, F., Flammini, A., Cieplak, M., and Banavar, J.R. (1996). Universality Classes of Optimal Channel Networks, *Science* 272, pp. 984–986 [A systematic study of the different classes of networks that arise for the optimal channel model.]

Mathias, N. and Gopal, V. (2001). Small Worlds: How and Why, *Physical Review E,* 63, 021117 [It shows how small world network can arise as a tradeoff between minimal wiring and maximal connectivity.]

Makse, H., Andrade, J.S., Batty, M., Havlin, S., Stanley, H.E. (1998) Modeling urban growth patterns with correlated percolation. *Phys. Rev. E* 58, 7054 [A model based on correlated percolation in presence of a gradient that describes the morphology of cities, the scaling of the urban perimeter of individual cities, and the area distribution of systems of cities.]

McMahon, T. A. and Bonner, J. T. (1983) *On Size and Life,* Scientific American Library, New York [Presents many of McMahon's models of animal locomotion in a very accessible form.]

Nash, J. (1950). Equilibrium points in n-person games, *Proceedings of the National Academy of Sciences* 36, pp. 48-49 [Gives the definition and conditions for existence for Nash equilibrium.]

Price, N.D., Reed, J.L., Palsson, B.O. (2004). Genome-scale models of microbial cells: evaluating the consequences of constraints. *Natural Review of Microbiology* 2, pp. 886–897 [It studies the effect of governing constrains on the functionality of empirically reconstructed metabolic networks.]

Runions, A., Fuhrer, A.M., Lane, B., Federl, P., Rolland-Lagan, A.-G., and Prusinkiewicz, P., (2005) Modeling and visualization of leaf venation patterns. *ACM Transactions on Graphics* 24(3): 702-711 [A method to model and visualize the venation pattern of leaves.]

Stevens, P.S. (1974). *Patterns in Nature*, Little, Brown, Boston [About golden mean, spirals, and general symmetries to be found in natural patterns.]

Valverde, S., Ferrer i Cancho R., and Solé R.V. (2002). Scale-free networks from optimal design, *Europhysics Letters*, 60, pp. 512-517 [Shows how small-world and scale-free properties of complex networks can arise from a local optimization procedure.]

Wardrop, J. G. (1952). *Some theoretical aspects of road traffic research*, Proceedings, Institute of Civil Engineers, PART II, 1, pp. 325-378. [The seminal paper of Wardrop that introduces the notion of Nash equilibrium for traffic.]

West, G.B., Brown, J.H. and Enquist, B.J. (1997). A general model for the origin of allometric scaling laws in biology, *Science* 276, pp. 122–126 [Provides a derivation of the 4/3 power in allometric scaling on the basis of a fractal model that describes the network that transports nutrients in living organisms.]

Youn, H., Gastner, M.T., and Jeong, H. (2008). Price of anarchy in transportation networks: efficiency and optimality control, *Physical Review Letter* 101: 128701 [Studies the Braess paradox for real transportation networks, showing how in different cities uncoordinated traffic leads to a larger global travel time.]

Biographical Sketches

Marc Barthelemy is a former student of the Ecole Normale Superieure of Paris. In 1992, he graduated at the University of Paris VI with a thesis titled "Random walks in random media". After his thesis, Marc Barthelemy focused on disordered systems and their properties. In 1999, he visited Prof. Stanley at Boston University and started to work on the properties of complex networks. Since 1992, he has held a position at the CEA (Paris) where his interests moved toward application of statistical physics to complex systems. In particular, he is now working in epidemiology and is interested in the modeling and characterization of complex systems and has co-authored more than 70 papers about disordered systems, networks, and epidemiology. He also co-authored (with A. Barrat and A. Vespignani) the book "Dynamical processes in complex networks" published by Cambridge University Press.

Alessandro Flammini received a PhD in Condensed Matter Physics form the International School for Advanced Studies (ISAS) in Trieste, Italy in 1996.His background is in statistical mechanics and its interdisciplinary applications. He matured his expertise in complex networks modeling through research appointments at MIT, the University of Cambridge, UK, and ISAS. He has co-authored 20+ papers about networks and their application to rivers, information sharing systems, protein interactions, roads, and general transportation systems. Flammini is currently an Assistant Professor in the School of Informatics, Faculty in the Complex Networks and Systems Center, and Adjunct Assistant Professor of Physics at Indiana University in Bloomington. He teaches courses on complex systems, probability and statistics, and logic.