DESCRIPTION OF CONTINUOUS LINEAR TIME-INVARIANT SYSTEMS IN FREQUENCY DOMAIN

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Keywords: Laplace transformation, Fourier transformation, Transfer function, Frequency-response, Bode-diagram, Bandwidth, Minimum and non-minimum phase systems, Differential equations, State-space description.

Contents

Summary

This article presents the most important descriptions of linear continuous time-invariant dynamical single input/single output (SISO) systems in the frequency domain. It starts with a short review of the Laplace transformation and shows how a single mathematical model for a system called a transfer function $G(s)$, can be found. This system representation allows the interconnection of subsystems in cascade, parallel or feedback form in an easy way. The interconnection between the transfer function model and a state space model is discussed. The introduction of the complex *G*-plane directly provides the definition of the frequency response, $G(\mathrm{j}\omega)$, as a special case of the

transfer function. Different ways for representing the frequency response, for example, in a polar plot and in a Bode-diagram, are considered. For the most common elementary dynamical systems, like P-, I-, D- and delay elements of first and second order, the behavior in the frequency domain is presented. Finally some important system characteristics like bandwidth and minimum as well as non-minimum phase behavior are defined.

1. Laplace Transformation

The Laplace transformation can be considered to be the most important aid for solving linear differential equations with constant coefficients. In most linear continuous-time control problems, the corresponding differential equations usually fulfil the necessary conditions for applying the Laplace transformation. The Laplace transformation represents an integral transformation that relates an original or time function $f(t)$ in a unique and invertible form to the Laplace or frequency function $F(s)$. Thus the unilateral Laplace transform of a time function or signal $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t\,, \tag{1}$$

where $f(t)$ is assumed to exist only for positive time, $0 < t \leq \infty$, and $s = \sigma + \mathrm{j}\omega$ is a complex-valued variable. For the existence of Eq. (1) the integral must converge. The smallest value of $\sigma_0 \leq \mathrm{Re}(s)$ for which this integral exists is called the abscissa of convergence.

The original time function can be recovered from $F(s)$ by the inversion integral

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi\mathrm{j}}\int_{c-\mathrm{j}\infty}^{c+\mathrm{j}\infty} F(s)\mathrm{e}^{st}\mathrm{d}s\,, \tag{2}$$

where $f(t) = 0$ for $t < 0$. Eq. (2) defines the inverse Laplace transform, where *c* is any real number greater than or equal to the abscissa of convergence. The path of integration is set parallel, with a displacement of *c*, to the $\mathrm{j}\omega -$axis in the *s*-plane. Furthermore, this path of integration is to the right of all singular points of $F(s)$.

It is important to note that the Laplace transform has properties of uniqueness and invertibility. Therefore, instead of using the inversion integral of Eq. (2) in many cases tables for Laplace transform pairs can directly be used, as for example Table 1.

No.	$f(t)$; $f(t) = 0$ for $t < 0$	$F(s)$
1	unit impulse $\delta(t)$	1

2	unit step $\sigma(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	t^2	$\frac{2}{s^3}$
5	$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	$t\,\mathrm{e}^{-at}$	$\frac{1}{(s+a)^2}$
8	$t^2\,\mathrm{e}^{-at}$	$\frac{2}{(s+a)^3}$
9	$t^n\,\mathrm{e}^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
10	$1-\mathrm{e}^{-at}$	$\frac{a}{s(s+a)}$
11	$\frac{1}{a^2}(\mathrm{e}^{-at}-1+at)$	$\frac{1}{s^2(s+a)}$
12	$(1-at)\,\mathrm{e}^{-at}$	$\frac{s}{(s+a)^2}$
13	$\sin\omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
14	$\cos\omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
15	$\mathrm{e}^{-at}\sin\omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$
16	$\mathrm{e}^{-at}\cos\omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$

Table 1. Laplace transform pairs

Linear differential equations with constant coefficients can be solved by Laplace transformation in three steps:

(i) Find the Laplace transform of the differential equation using Table 1 as well as the properties of Laplace transformation listed in Table 2.

(ii) Solve the algebraic equation in the *s*-domain for $F(s)$.

(iii) Obtain $f(t)$ by the inverse Laplace transform of $F(s)$.

Example:

Given the differential equation

$$\ddot{f}(t) + 3\dot{f}(t) + 2f(t) = \mathrm{e}^{-t}$$

with the initial conditions $f(0+) = \dot{f}(0+) = 0$, its solution can be found in the three steps given above:

(i) $s^2 F(s) + 3s F(s) + 2F(s) = \dfrac{1}{s+1}$

(ii) $F(s) = \dfrac{1}{s+1} \cdot \dfrac{1}{s^2 + 3s + 2}$

(iii) Before applying the inverse Laplace transformation, $F(s)$ is expanded into partial fractions as follows:

$$F(s) = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}.$$

Taking the cases No. 6 and 7 in Table 1 and the properties of Laplace transforms in Table 2 into account the inverse Laplace transform of $F(s)$ directly gives the solution of the above differential equation in the form

$$f(t) = \mathrm{e}^{-2t} - \mathrm{e}^{-t} + t\mathrm{e}^{-t}.$$

This example shows the importance of the poles s_1, s_2 and s_3 of $F(s)$ in the dynamic behavior, discussed in more detail in section 3.2. Since all poles here have a negative real part, $\lim_{t\to\infty} f(t) = 0$. If the real part of any pole were positive, the final value of $f(t)$ would tend to infinity. As in control problems, $f(t)$ usually represents a physical signal, the dynamic behavior of such a control system can be directly analyzed from the pole location of $F(s)$. This shows how highly important the pole locations of $F(s)$ are for stability.

1	$\mathcal{L}\{af(t)\} = aF(s)$
2	$\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)$
3	$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$
4	$\mathcal{L}\{f(t-a)\} = \mathrm{e}^{-as} F(s)$

5	$\mathcal{L}\left\{\frac{\mathrm{d}\,f(t)}{\mathrm{d}t}\right\} = sF(s) - f(0+)$
6	$\mathcal{L}\left\{\frac{\mathrm{d}^2\,f(t)}{\mathrm{d}t^2}\right\} = s^2F(s) - s\,f(0+) - \frac{\mathrm{d}}{\mathrm{d}\,t}\,f(0+)$
7	$\mathcal{L}\left\{\int\limits_0^t f(\tau)\,\mathrm{d}\tau\right\} = \frac{1}{s}F(s)$
8	$\mathcal{L}\left\{\int\limits_0^t f_1(\tau)\,f_2(t-\tau)\,\mathrm{d}\tau\right\} = F_1(s)\,F_2(s)$
9	$\mathcal{L}\,\{f_1(t)\,f_2(t)\} = \frac{1}{2\pi\mathrm{j}}\int\limits_{c-\mathrm{j}\infty}^{c+\mathrm{j}\infty} F_1(p)\,F_2(s-p)\,\mathrm{d}p$
10	$f(0+) = \lim\limits_{s\to\infty} s\,F(s)$
11	$f(\infty) = \lim\limits_{s\to 0} s\,F(s)$

Table 2. Properties of the Laplace transformation

2. Fourier Transformation

If a signal $f(t)$ is considered over the time-domain $-\infty \le t \le +\infty$, then the Fourier transform

$$\mathcal{L}\{f(t)\} = F(\mathrm{j}\omega) = \int\limits_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \qquad (3)$$

and the inverse Fourier-transform

$$\mathcal{F}^{-1}\{F(\mathrm{j}\omega)\} = f(t)$$
$$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F(\mathrm{j}\omega)\,\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega \qquad (4)$$

The complex-valued Fourier transform can be written as

$$F(\mathrm{j}\omega) = R'(\omega) + \mathrm{j}I'(\omega)$$

or

$$F(\mathrm{j}\omega) = A'(\omega)\,\mathrm{e}^{\mathrm{j}\varphi'(\omega)},$$

where $A'(\omega) = |F(\mathrm{j}\omega)| = \sqrt{R'^2(\omega) + I'^2(\omega)}$ is the Fourier spectrum or amplitude density spectrum and $\varphi'(\omega) = \arctan[I'(\omega)/R'(\omega)]$ is the phase angle curve of $F(\mathrm{j}\omega)$. The Fourier transform of $f(t)$ only exists if $\int_{-\infty}^{\infty} |f(t)|\mathrm{d}t < \infty$.

3. Transfer Function of a Dynamical System

3.1. Definition

Continuous-time, linear time-invariant systems with lumped parameters and without transportation lag having a **s**ingle **i**nput $x_\mathrm{i}(t)$ and a **s**ingle **o**utput $x_0(t)$ (also known as SISO systems) can be described by ordinary differential equations of the general form

$$\sum_{i=0}^{n} a_i \frac{\mathrm{d}^i x_0(t)}{\mathrm{d}t^i} = \sum_{j=0}^{m} b_j \frac{\mathrm{d}^j x_\mathrm{i}(t)}{\mathrm{d}t^j}, \quad m \le n. \tag{5}$$

The transfer function $G(s)$ of this system is defined to be the ratio of the Laplace transforms of the output (or response function) $x_0(t)$ and the input (or driving function) $x_\mathrm{i}(t)$ under the assumption that all initial conditions are zero:

$$\begin{aligned} G(s) &= \frac{\mathrm{L}\{x_0(t)\}}{\mathrm{L}\{x_\mathrm{i}(t)\}} = \frac{X_0(s)}{X_\mathrm{i}(s)} \\ &= \frac{b_0 + b_1 s + \ldots + b_m s^m}{a_0 + a_1 s + \ldots + a_n s^n}. \end{aligned} \tag{6}$$

The transfer function is a property of the system itself, and is therefore independent of $x_\mathrm{i}(t)$. If a system exhibits a pure delay the input and output of such a dead-time or delay element are related by

$$x_0(t) = x_\mathrm{i}(t - T_\mathrm{d}), \tag{7}$$

where T_d is the dead-time. The transfer function of such a system is obtained by applying property 4 of Table 2 to the equation above

$$G_\mathrm{d}(s) = \frac{\mathrm{L}\{x_0(t)\}}{\mathrm{L}\{x_\mathrm{i}(t)\}} = \mathrm{e}^{-sT_\mathrm{d}}. \tag{8}$$

If a system according to Eq. (5) or Eq. (6) contains an additional dead-time element, the differential equation changes to

$$\sum_{i=0}^{n} a_i \frac{\mathrm{d}^i x_0(t)}{\mathrm{d}t^i} = \sum_{j=0}^{m} b_j \frac{\mathrm{d}^j x_\mathrm{i}(t - T_\mathrm{d})}{\mathrm{d}t^j} \tag{9}$$

and the corresponding transfer function becomes

$$\begin{aligned} G(s) &= \frac{b_0 + b_1 s + \ldots + b_m s^m}{a_0 + a_1 s + \ldots + a_n s^n} \mathrm{e}^{-sT_\mathrm{d}} \\ &= \frac{B(s)}{A(s)} \mathrm{e}^{-sT_\mathrm{d}} . \end{aligned} \tag{10}$$

The response of a linear system to a unit impulse $\delta(t)$ is the so-called impulse-response or weighting function $g(t)$. Because of the fact that $\mathcal{L}\{\delta(t)\} = 1$ (see Table 1, No. 1) and Eq. (6) we have

$$G(s) = \frac{X_0(s)}{X_\mathrm{i}(s)} = \frac{\mathcal{L}\{g(t)\}}{\mathcal{L}\{\delta(t)\}} = \mathcal{L}\{g(t)\} . \tag{11}$$

Hence, the transfer function $G(s)$ can also be defined as the Laplace transform of $g(t)$. Therefore $G(s)$ and $g(t)$ contain the same and complete information about the system dynamics.

3.2. Poles and Zeros of $G(s)$

It often seems reasonable to factorize the rational transfer function of Eq. (6) as follows:

$$\begin{aligned} G(s) &= \frac{B(s)}{A(s)} \\ &= k_0 \frac{(s - s_{\mathrm{z}_1})(s - s_{\mathrm{z}_2}) \ldots (s - s_{\mathrm{z}_m})}{(s - s_{\mathrm{p}_1})(s - s_{\mathrm{p}_2}) \ldots (s - s_{\mathrm{p}_n})} \end{aligned} \tag{12}$$

Due to physical reasons, the coefficients a_i and b_j in Eq. (6) are always real. However, the zeros s_{z_j} and poles s_{p_i} can be real or conjugate complex-valued. Poles and zeros can be easily represented in the complex *s*-plane as depicted in Figure 1. A linear time-invariant system without dead-time is thus completely characterized by the location of its poles and zeros, and the factor k_0. Furthermore, the poles of $G(s)$ are of specific importance. To show this we consider the undisturbed

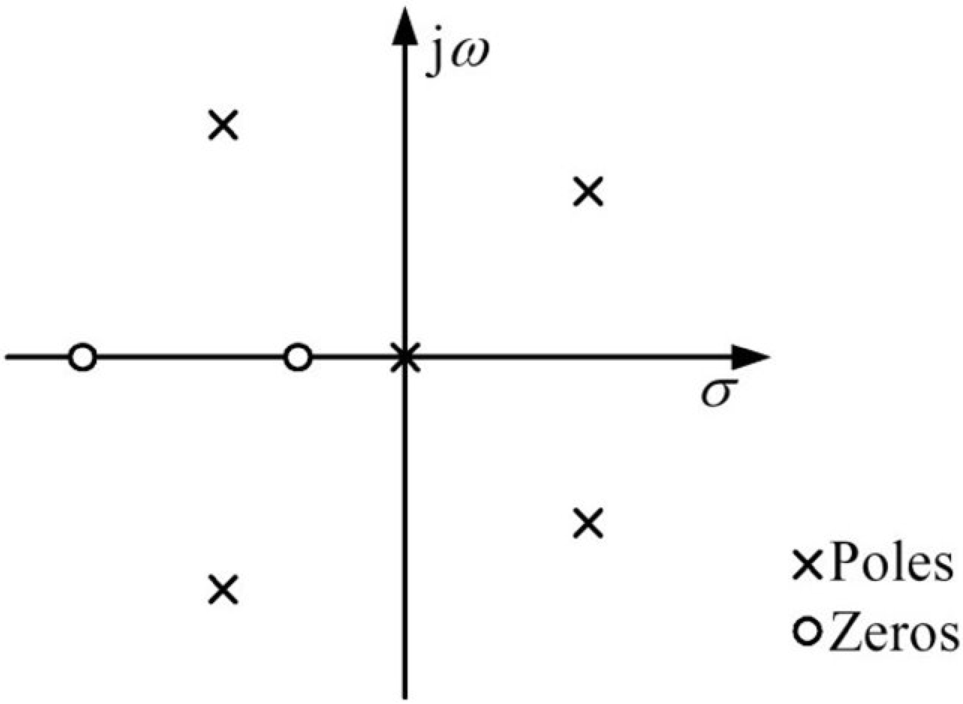

Figure 1. Location of the poles and zeros of a system in the *s*-plane

system in Eq. (5), where $x_\mathrm{i}(t) \equiv 0$ and only *n* initial conditions are giving rise to the output $x_0(t)$. This leads to the problem of solving the homogeneous differential equation

$$\sum_{i=0}^{n} a_i \frac{\mathrm{d}^i x_0(t)}{\mathrm{d}t^i} = 0.$$

Substituting $x_0(t) = \mathrm{e}^{st}$ we get the characteristic equation

$$\sum_{i=0}^{n} a_i s^i = 0, \tag{13}$$

which can also be obtained from Eq. (6) or Eq. (12) by setting the denominator polynomial $A(s) = 0$, under the assumption that $A(s)$ and $B(s)$ have no common roots. It should be noted that the solutions of the characteristic equation are identical to the poles of the corresponding transfer function.

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Biographical Sketch

Heinz Unbehauen is Professor Emeritus at the Faculty of Electrical Engineering and Information Sciences at Ruhr-University, Bochum, Germany. He received the Dipl.-Ing. degree from the University of Stuttgart, Germany, in 1961 and the Dr.-Ing. and Dr.-Ing. habil. degrees in Automatic Control from the same university in 1964 and 1969, respectively. In 1969, he was awarded the title of Docent and in 1972, he was appointed as professor of control engineering in the Department of Energy Systems at the University of Stuttgart. Since 1975, he has been Professor at Ruhr-University of Bochum, Faculty of Electrical Engineering, where he was head of the Control Engineering Laboratory until February 2001. He was Dean of his faculty in 1978/79. He was a Visiting Professor in Japan, India, China and the USA. He has authored and co-authored over 400 journal articles, conference papers and 7 books. He has delivered many invited lectures and special courses at universities and companies around the world. His main research interests are in the fields of system identification, adaptive control, robust control and process control of multivariable systems. He is Honorary Editor of IEE Proceedings on Control Theory and Application and System Science, Associate Editor of Automatica and serves on the Editorial Board of the International Journal of Adaptive Control and Signal Processing, Optimal Control Applications and Methods (OCAM) and Systems Science. He also served as associate editor of IEEE-Transactions on Circuits and Systems as well as Control-Theory and Advanced Technology (C-TAT). He is also an Honorary Professor of Tongji University Shanghai. He has been a consultant for many companies as well as for public organisations, e.g., UNIDO and UNESCO. He is member of several national and international professional organisations and Fellow of IEEE.