OBSERVER DESIGN

Boris Lohmann
Institut für Automatisierungstechnik, Universität Bremen, Germany

Keywords: state observer, state estimator, Luenberger observer, SISO Systems, pole placement, reduced order observer.

Contents

Summary

The technical realization of state feedback controllers requires that all of the n state variables are accessible to measurement and feedback. This is not always the case due to cost and availability of measurement equipment or for other reasons. The problem can be overcome by introducing a so-called *state estimator* or *state observer*, generating an estimate of the state $\mathbf{x}$ from only one measured system variable and the accessible control input u.

This state estimate is then used for operating state feedback controllers as introduced in *Controller Design*. It will turn out that integrating the observer into the closed-loop system does not shift the eigenvalues away from the locations originally specified, i.e. state observer and state feedback can be designed separately. Thereby, the combination of state observer and state feedback become a powerful tool of linear control-system design.

1. Objectives and Structure of the State Observer

Consider the state space description of a linear time-invariant dynamic system with one scalar control input $u(t)$ and the $(n,1)$-state vector $\mathbf{x}(t)$,

State differential equation: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$, (1)

and the,

Initial state: $\mathbf{x}(t_0) = \mathbf{x}_0$. (2)

The design of state feedback controllers as introduced in *Controller Design* assumes that all the elements of the state vector are at our disposal. If this is not the case, one may think of estimating instead of measuring the state. This idea is investigated in the following sections with the objective of generating a *state estimate* from only one system variable accessible to measurement. This *measurement output* variable is defined by

Measurement equation: $y(t) = \mathbf{c}^{\mathrm{T}}\mathbf{x}(t)$, (3)

i.e., as a linear combination of the states. In many cases the measured output is identical to the control output y as used in *Description and Analysis of Dynamic Systems in State Space*, and *Controller Design*. Otherwise, in order to avoid confusion, we will add a subscript c to the *control output* and define it by

Output equation: $y_{\mathrm{c}}(t) = \mathbf{c}_{\mathrm{c}}^{\mathrm{T}}\mathbf{x}(t)$. (4)

A simple approach to the estimation of the system state is shown in Figure 1: a model of the plant is installed in parallel to the control plant, where the model $\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{b}u(t)$ is driven by the same control input signal u as the real control plant. Provided both systems start from the same initial state, we may hope that they will follow the same trajectories, i.e. $\hat{\mathbf{x}}(t) = \mathbf{x}(t)$.

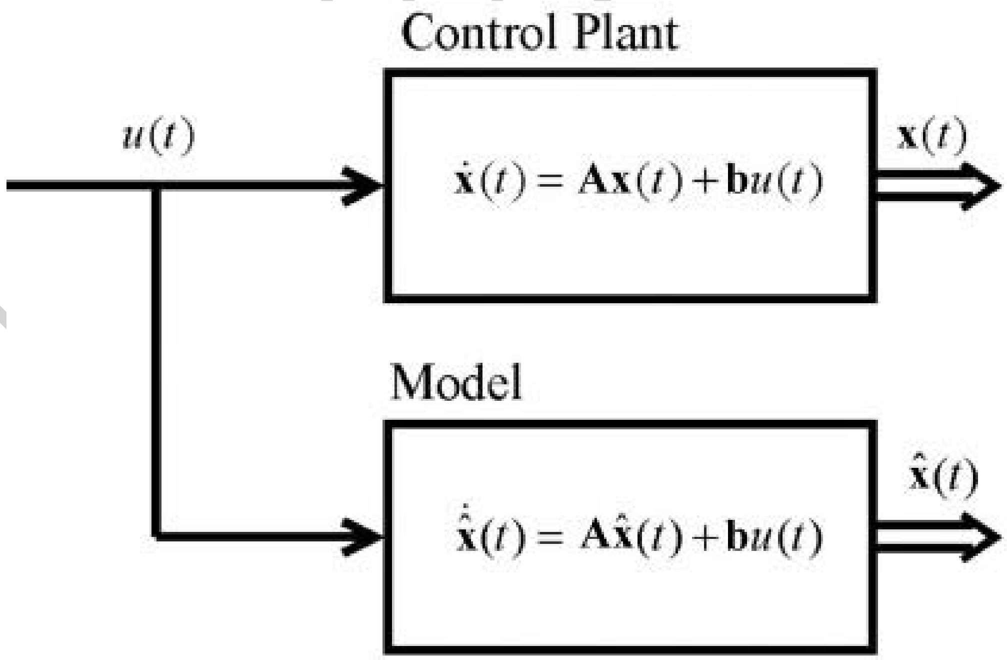

Figure 1: Simple approach to state estimation by a parallel model

In practice however, the initial state will not be known. Therefore the structure of the observer has to be extended. Following an idea of *D.G. Luenberger* this is done by comparing the measurement output $y(t) = \mathbf{c}^{\mathrm{T}}\mathbf{x}(t)$ with the corresponding variable $\hat{y}(t) = \mathbf{c}^{\mathrm{T}}\hat{\mathbf{x}}(t)$ (generated from the state estimate $\hat{\mathbf{x}}$) and by feeding back this difference into the model with an appropriate gain **l**. The resulting structure is shown in Figure 2 and is called *state estimator, state observer*, or *Luenberger state observer*.

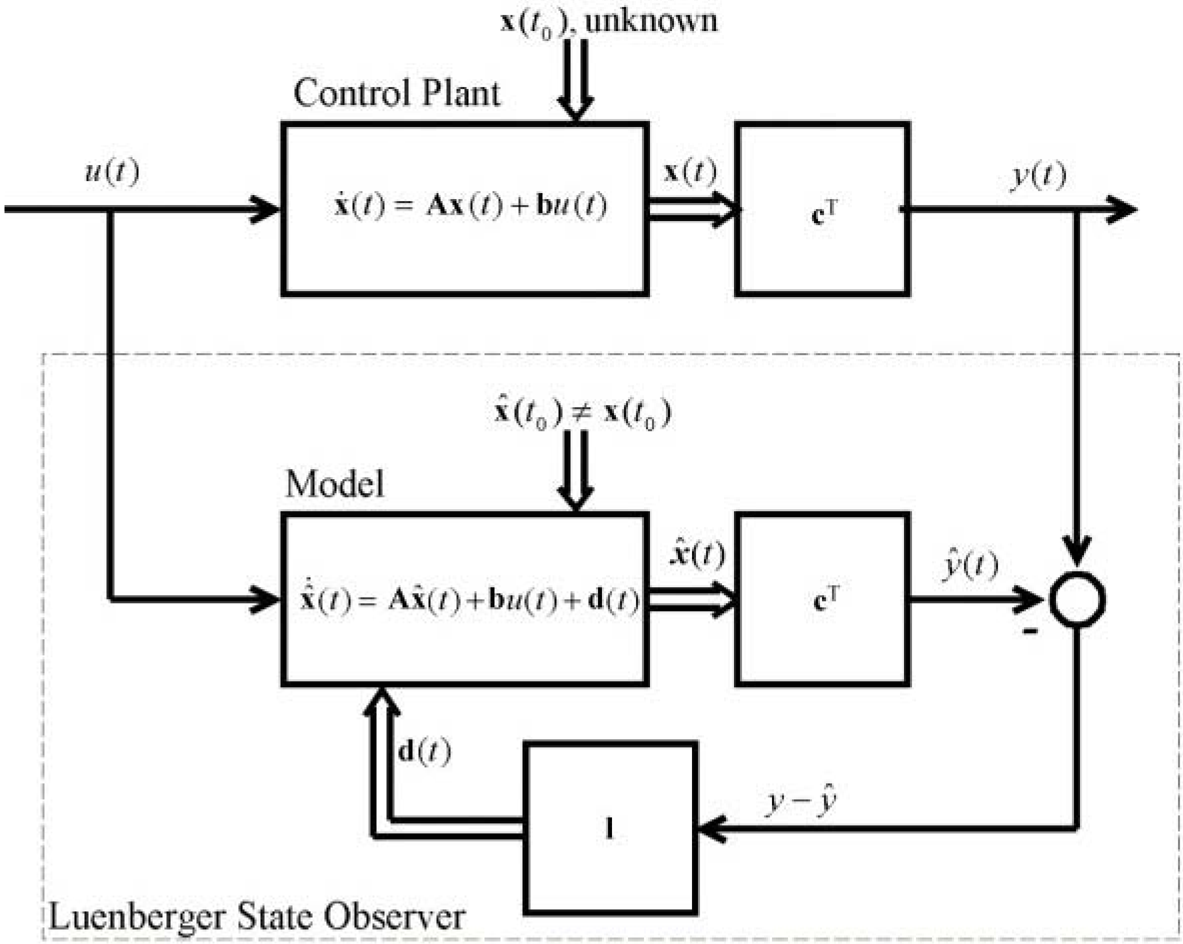

Figure 2: Structure of a state observer

The difference $y - \hat{y}$ is fed back directly into the model via a vector **l** of gain factors $l_1, ..., l_n$. Thus, the state differential equation of the parallel model is

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{b}u(t) + \mathbf{l}\left(y(t) - \hat{y}(t)\right), \tag{5}$$

and by substituting

$$\hat{y}(t) = \mathbf{c}^{\mathrm{T}}\hat{\mathbf{x}}(t), \tag{6}$$

we obtain

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{l}\mathbf{c}^{\mathrm{T}})\hat{\mathbf{x}}(t) + \mathbf{b}u(t) + \mathbf{l}y(t). \tag{7}$$

The problem remaining is to select the gain vector **l** so that the state estimate $\hat{\mathbf{x}}(t)$ converges to the state vector $\mathbf{x}(t)$. We summarize:

A ***state observer*** for the plant $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$ with the *measurement output* $y(t) = \mathbf{c}^{\mathrm{T}}\mathbf{x}(t)$ is set up as

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{l}\mathbf{c}^{\mathrm{T}})\hat{\mathbf{x}}(t) + \mathbf{b}u(t) + \mathbf{l}y(t)\,. \tag{8}$$

The design task is to select the gain vector **l** so that the ***estimation error***

$$\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t) \tag{9}$$

converges to zero as $t \to \infty$ for any initial states $\mathbf{x}(t_0), \hat{\mathbf{x}}(t_0)$.

If we succeed in finding an appropriate **l**, we can use state $\hat{\mathbf{x}}$ as an estimate of the real state vector **x** for a variety of purposes and in particular for operating a state feedback controller.

2. Design of the Observer

In order to find out under what conditions the estimation error $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ decays, its time derivative is determined,

$$\dot{\tilde{\mathbf{x}}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\mathbf{x} + \mathbf{b}u - (\mathbf{A} - \mathbf{l}\mathbf{c}^{\mathrm{T}})\hat{\mathbf{x}} - \mathbf{b}u - \mathbf{l}y\,.$$

With $y(t) = \mathbf{c}^{\mathrm{T}}\mathbf{x}(t)$, this equation simplifies to

$$\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{l}\mathbf{c}^{\mathrm{T}})(\mathbf{x} - \hat{\mathbf{x}}) = (\mathbf{A} - \mathbf{l}\mathbf{c}^{\mathrm{T}})\tilde{\mathbf{x}}\,, \tag{10}$$

which is a differential equation of $\tilde{\mathbf{x}}$, the so-called *state error differential equation*. It is a homogeneous state differential equation with the solution

$$\tilde{\mathbf{x}}(t) = e^{(\mathbf{A} - \mathbf{l}\mathbf{c}^{\mathrm{T}})(t - t_0)}\tilde{\mathbf{x}}(t_0)\,. \tag{11}$$

Its stability properties can easily be examined: *The estimation error* $\tilde{\mathbf{x}}$ *will converge to zero if and only if all eigenvalues of* $(\mathbf{A} - \mathbf{l}\mathbf{c}^{\mathrm{T}})$ *- the so-called observer eigenvalues - are located in the left half of the complex plane. The observer is then called stable.* The objective of the control-system engineer is to select **l** so that $(\mathbf{A} - \mathbf{l}\mathbf{c}^{\mathrm{T}})$ has stable eigenvalues in order for $\tilde{\mathbf{x}}$ to decay to zero.

By analogy to state feedback design, we will specify the eigenvalues of $(\mathbf{A} - \mathbf{l}\mathbf{c}^{\mathrm{T}})$ and then determine the appropriate vector **l**. The design steps are substantially the same as in the design of state feedback controllers by pole placement. This analogy can be expected at a glance when comparing the matrix $(\mathbf{A} - \mathbf{l}\mathbf{c}^{\mathrm{T}})$ with the system matrix

$(\mathbf{A}-\mathbf{bk}^{\mathrm{T}})$ of a state feedback controlled system.

2.1. Observer Design by Matching of Coefficients

The control-system designer specifies n eigenvalues $\beta_1,...,\beta_n$. In order to make them eigenvalues of the matrix $(\mathbf{A}-\mathbf{lc}^{\mathrm{T}})$, the vector $\mathbf{l}$ is to be found so that the characteristic equation of $(\mathbf{A}-\mathbf{lc}^{\mathrm{T}})$ has the roots $\beta_1,...,\beta_n$, i.e.

$$\det(s\mathbf{I}-\mathbf{A}+\mathbf{lc}^{\mathrm{T}})=(s-\beta_1)\cdot\ldots\cdot(s-\beta_n)\,. \tag{12}$$

By calculating the determinant and expanding the right hand side, we obtain

$$s^n+a_{n-1}(\mathbf{l})\cdot s^{n-1}+\ldots+a_0(\mathbf{l})=s^n+p_{n-1}s^{n-1}+\ldots+p_0\,, \tag{13}$$

where the coefficients a_ν are functions of the elements $l_1,...,l_n$ of $\mathbf{l}$. By setting like coefficients equal to each others, the n equations

$$\begin{aligned} a_{n-1}(\mathbf{l}) &= p_{n-1}, \\ &\vdots \\ a_0(\mathbf{l}) &= p_0. \end{aligned}$$

are obtained. In fact, they are linear in the n unknowns and can be solved for $l_1,...,l_n$ easily. This determines the observer.

The observer gain vector

$$\mathbf{l}=\begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix}, \tag{14}$$

found by solving the n linear equations

$$\begin{aligned} a_{n-1}(\mathbf{l}) &= p_{n-1}, \\ &\vdots \\ a_0(\mathbf{l}) &= p_0, \end{aligned} \tag{15}$$

places the eigenvalues of $(\mathbf{A}-\mathbf{lc}^{\mathrm{T}})$ in the desired locations $\beta_1,...,\beta_n$.
$a_\nu(\mathbf{l})$ are the coefficients of the characteristic polynomial $\det(s\mathbf{I}-\mathbf{A}+\mathbf{lc}^{\mathrm{T}})$,
p_ν are the coefficients of the desired polynomial

$$p(s)=(s-\beta_1)\cdot\ldots\cdot(s-\beta_n)=s^n+p_{n-1}s^{n-1}+\ldots+p_0\,.$$

-
-
-

Bibliography

Ackermann J. (1972). Der Entwurf linearer Regelungssysteme im Zustandsraum. *Regelungstechnik* **20**, 297-300. [Presents Ackermann's formula]

DeCarlo R.A. (1989). *Linear Systems*. Prentice Hall. [Includes an introduction to the description and analysis by state space methods].

Franklin G.F., Powell J.D. and Emami-Naeini A. (1994). *Feedback Control of Dynamic Systems*, 3rd edition. Addison-Wesley. [Textbook with good introduction to the key objectives of automatic control. Mathematical appendix].

Gilbert E.G., (1963). Controllability and Observability in Multivariable Control Systems. *J. Control, Ser. A*, **1**,. 128-151. [Includes the controllability/observability criteria presented].

Hautus M.L.J. (1969). Controllability and Observability Conditions of Linear Autonomous Systems. *Indagationes Mathematicae* **31**, 443-448. . [Includes the controllability/observability criteria presented].

Kaczorek T. (1992). *Linear Control Systems. Volume 1 and 2*. Research Studies Press. [Extensive theoretical work; includes linear differential algebraic systems and many details].

Kailath, T. (1980). *Linear Systems*. Prentice Hall. [A standard textbook on linear theory].

Kalman, R.E. (1960). A new Approach to Linear Filtering and Prediction Problems. *J. Basic Engineering* **85**, 394-400. [An initial work on state space methods].

Kalman R.E. (1960). *On the General Theory of Control Systems*. Proc. 1st Int. Congr. Autom. Control, Moscow, pp. 481-492. [An initial work on state space methods].

Kuo B.C. (1995). *Automatic Control Systems*, 7th edition. Prentice Hall. [Extensive textbook. Some details missing in favor of good readability].

Luenberger D.G. (1964). Observing the State of a Linear System. *IEEE Transactions on Military Electronics* **8**, 74-80. [Presents Luenberger's approach to state observers].

Luenberger D.G. (1966). Observers for Multivariable Systems. *IEEE Transactions on Automatic Control* **11**, 190-197. [Presents Luenberger's approach to state observers].

Luenberger D.G. (1971). An Introduction to Observers. *IEEE Transactions on Automatic Control* **16**, 596-602. [Presents Luenberger's approach to state observers].

MacFarlane A.G.J. and Karcanias N. (1976). Poles and Zeroes of linear multivariable systems: a survey of the algebraic, geometric and complex-variable theory. *Int. J. Control* **24**, 33-74. [Introduces the concept of invariant zeros from a state variable approach].

Shinners S.M. (1998). *Advanced Modern Control System Theory and Design*. John Wiley & Sons. [A recent textbook].

MATLAB applications: The following two web-addresses provide introductory examples on how to use the software package MATLAB for control system design purposes: http://tech.buffalostate.edu/ctm/ and http://www.ee.usyd.edu.au/tutorials_online/matlab/index.html

Biographical Sketch

Boris Lohmann received the Dipl.-Ing. and Dr.-Ing. degrees in electrical engineering from the Technical University of Karlsruhe, Germany, in 1987 and 1991 respectively. From 1987 to 1991 he was with the Fraunhofer Institut (IITB) and with the Institute of Control Systems, Karlsruhe, working in the fields of autonomous vehicles control and multi-variable state space design.

From 1991 to 1997 he was with AEG Electrocom Automation Systems in the development department for postal sorting machines, at last as the head of mechanical development. In 1994 he received the 'Habilitation' degree in the field of system dynamics and control from the Universität der Bundeswehr, Hamburg, for his results on model order reduction of nonlinear dynamic systems.

Since 1997 he has been full professor at the University of Bremen, Germany, and head of the Institute of Automation Systems. His fields of research include nonlinear multivariable control theory; system modeling, simplification, and simulation; and image-based control systems, with industrial applications in the fields of autonomous vehicle navigation, active noise reduction, error detection and fault diagnosis.