MEASUREMENTS OF FREQUENCY RESPONSE FUNCTIONS

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Contents

Summary

Frequency response function (FRF) measurements are an interesting intermediate step in the identification process. The complexity of the modeling problem is visualized before starting the parametric modeling, the quality of the measurements is assessed in an early phase. In this chapter a number of basic and advanced FRF measurement methods are discussed. An analysis of the bias and efficiency of the FRF measurements is made, and their dependency on the experimental conditions and on the excitation signal is analyzed. Simple and more advanced averaging techniques are proposed to improve the quality of the FRF measurement. Guidelines given at the critical steps of the FRF measurement process enable the less experienced user to start modeling from good raw data.

1. Introduction

Consider the linear dynamic system G between the input $u(t)$ and the output $y(t)$ as shown in Figure 1. The aim of the following chapters in this topic is to build a parametric model for this system,

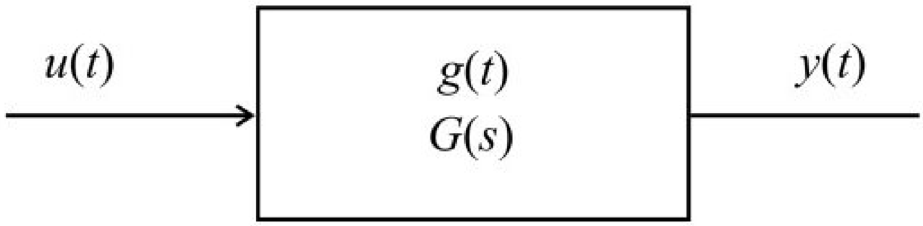

Figure 1. Block diagram of the system

identifying, for example, a transfer function $G(s,\theta)$ or $G(z,\theta)$. Such a model is called a parametric model since it employs a finite-dimensional parameter vector. Parametric modeling requires a series of user decisions (e.g. selection of the order of numerator and denominator of $G(s)$), thus it is strongly advised to get a good initial idea about the system under test. Step or impulse response measurements provide this information. Also frequency response (FRF) measurements are very valuable. An FRF consists of the measurement of the transfer function $G(s = j\omega_k)$ at a discrete set of frequencies $\omega_k, k = 1,...,F$. All these models are called non-parametric since the information is not condensed into a small set of parameters. In this chapter we focus, exclusively, on FRF measurements. A series of basic questions is addressed:

- How are the bias and efficiency of the FRF measurements influenced by the experimental conditions?
- How should the excitation signal be chosen?
- Can we improve the quality of the FRF using averaging methods?
- Can we quantify the quality of the FRF-measurements?

All these aspects are discussed below. Starting from a straightforward solution, the more advanced techniques are introduced step by step, showing each time what additional problems are addressed by these more advanced techniques. Since FRF measurement techniques heavily rely on the time-to-frequency domain transformation of sampled signals, we will spend some time on the most important aspects of the discrete Fourier transform.

2. An Introduction to the Discrete Fourier Transform

In most situations, real life systems are naturally continuous in time. However, most signal processing is done, nowadays, on digital computers that operate on discrete-time signals. In practice the continuous-time signals are discretized (sampled) and quantized (digitized) so that the signal can finally be stored in the memory of a digital computer. Next, the spectrum of these signals is needed in order to calculate the FRF of the system. This is done using the discrete Fourier transform (DFT), usually calculated with the fast Fourier transform algorithm (FFT). Each of these steps creates errors, and it is important for a user to understand their behavior to minimize the impact of the errors on the results. In this section only a brief introduction is given. For an extended overview, the reader is referred to the literature. First we discuss, briefly, the sampling process, next we show how to "measure" the Fourier spectrum of a signal, and finally we focus on the spectral properties of periodic excitations and how to exploit them to minimize the measurement errors.

2.1. The Sampling Process

The continuous-time signal is sampled uniformly with sampling interval T_s, and is represented by the equivalent discrete-time sequence $u_\mathrm{d}(n) = u(nT_\mathrm{s})$, $N = -\infty, \ldots, -1, 0, 1, \ldots, +\infty$. In the time domain, the sampling process can be formulated as a multiplication with a periodically repeated Dirac impulse:

$$\tilde{u}_\mathrm{d}(t) = u(t)\delta_{T_\mathrm{s}}(t) \text{ with } \delta_{T_\mathrm{s}}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_\mathrm{s}) \tag{1}$$

Note that in this framework the discrete-time signal $u_\mathrm{d}(n)$ is formally represented by a continuous-time signal $\tilde{u}_\mathrm{d}(t)$ that carries all its power at the discrete-time instances nT_s. Define the spectrum of the discrete-time signal as

$$U_\mathrm{d}\left(e^{j2\pi fT_\mathrm{s}}\right) = \sum_{n=-\infty}^{\infty} u_\mathrm{d}(n) e^{-j2\pi fnT_\mathrm{s}} \tag{2}$$

Then the following relation exists:

$$U_\mathrm{d}\left(e^{j2\pi fT_\mathrm{s}}\right)=\tilde{U}_\mathrm{d}\left(j2\pi f\right)=F\left\{\tilde{u}_d\left(t\right)\right\}=\int_{-\infty}^{\infty}\tilde{u}_\mathrm{d}\left(t\right)e^{-j2\pi ft}dt\ . \qquad (3)$$

The spectrum $U_\mathrm{d}\left(e^{j\omega T_\mathrm{s}}\right)$ is linked to $U\left(j\omega\right)$ by noticing that the multiplication in the time domain, $u\left(t\right)\delta_{T_\mathrm{s}}\left(t\right)$, corresponds to the convolution of the spectra in the frequency domain, $U\left(j2\pi f\right)*\left(f_\mathrm{s}\delta_{f_\mathrm{s}}\left(f\right)\right)$, with $f_\mathrm{s}\delta_{f_\mathrm{s}}\left(f\right)$ the spectrum of $\delta_{T_\mathrm{s}}\left(t\right)$, and $\delta_{f_\mathrm{s}}\left(f\right)$ a periodically repeated Dirac impulse with period $f_\mathrm{s}=1/T_\mathrm{s}$

$$\delta_{f_\mathrm{s}}\left(f\right)=\sum_{k=-\infty}^{+\infty}\delta\left(f-kf_\mathrm{s}\right) \qquad (4)$$

Using (4), we get

$$U_d\left(e^{j2\pi fT_\mathrm{s}}\right)=U\left(j2\pi f\right)*\left(f_\mathrm{s}\delta_{f_\mathrm{s}}\left(f\right)\right)=\frac{1}{T_\mathrm{s}}\sum_{k=-\infty}^{+\infty}U\left(j2\pi\left(f-kf_\mathrm{s}\right)\right). \qquad (5)$$

The convolution of the spectra is illustrated in Figure 2.

It shows that the sampling process results in a repeated spectrum in the frequency domain with period f_s. If the bandwidth f_B of the sampled signal is larger than half the sampling frequency, the shifted spectra overlap and information is lost. Therefore, it is important to restrict the bandwidth below half the sampling frequency $f_B<f_\mathrm{s}/2$ in order to avoid errors. This error is called the aliasing error and the condition on the sample frequency is known as Shannon's sampling theorem. In practice it is often necessary to put anti-alias filters to eliminate the high frequency spectral content of the signal.

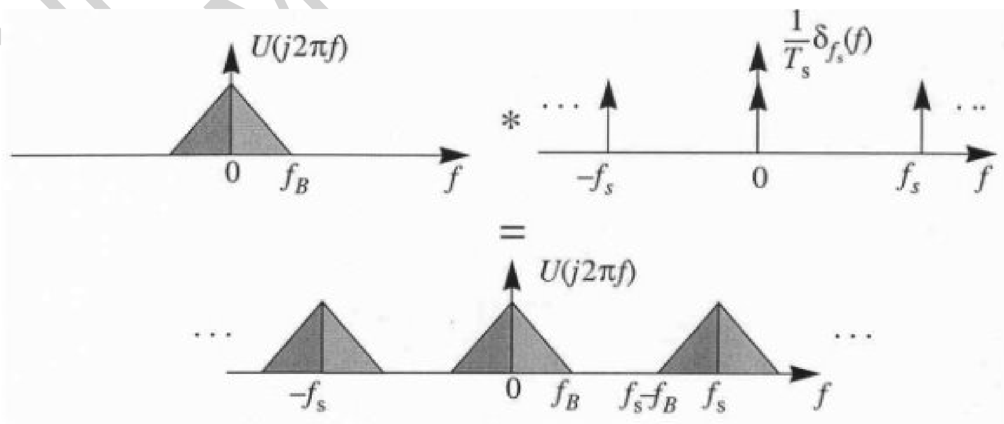

Figure 2. Impact of the domain discretization (sampling) on the spectrum.

2.2. The Discrete Fourier Transform (DFT-FFT)

Three basic steps have to be made to measure the spectrum of a continuous-time signal:

- Discretization in time: sample the continuous-time signal at an equidistant time grid.
- Restrict the length of the data record: our computers can only deal with a finite number of data. Thus the length of the record is restricted to N samples, excluding the rest. This is called windowing.
- Discretization in frequency: the finite length discrete-time signal has still a continuous frequency spectrum. The value of this spectrum will only be calculated at an equidistant set of frequencies.

The impact of all these steps is illustrated in more detail below, in a simple example. The continuous-time signal $u(t) = \cos 2\pi f_0 t$, with $f_0 = 5.5$ Hz is sampled at $f_s = 64$ Hz during 1 second. From these measurements we will calculate the discrete Fourier transform step by step.

2.2.1. Discretization in Time

The sampling process has already been discussed in the previous section. Figure 3 shows the signal together with its spectrum before and after sampling. In order to keep enough detail in the shown figures, a zoom is made in the frequency band [-10 Hz, 10 Hz]. The periodic repetitions of the spectrum of the discrete-time sequence are not shown. Note that if no aliasing appears, the spectrum of the continuous-time and the discrete-time signal are equal to each other within a scale factor.

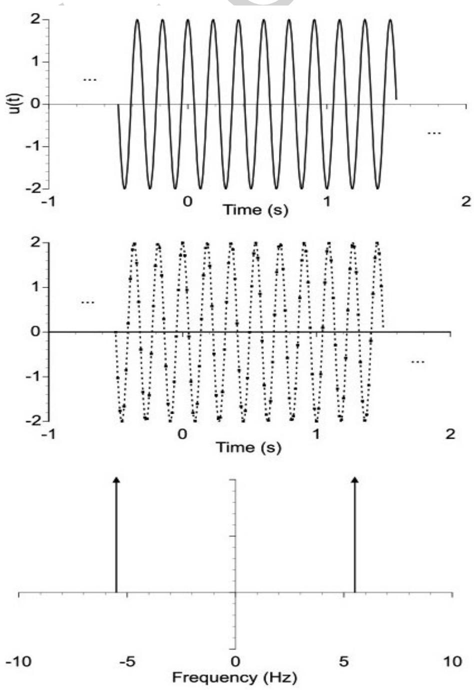

Figure 3. The time signal before and after sampling together with the spectrum in the frequency band [-10 Hz, 10 Hz].

Mathematical operation:

time domain:

$$\tilde{u}_{\mathrm{d}}(t) = \sum_{n=-\infty}^{+\infty} u(t)\delta\left(t - nT_{\mathrm{s}}\right)$$

frequency domain:

$$U_{\mathrm{d}}\left(e^{j2\pi f T_{\mathrm{s}}}\right) = \frac{1}{T_{\mathrm{s}}} \sum_{k=-\infty}^{+\infty} U\left(j2\pi\left(f - kf_{\mathrm{s}}\right)\right) \tag{6}$$

2.2.2. Windowing

The sampled signal has still an infinite length ($]-\infty,\infty[$). Since the computer can only process a finite number of samples, we have to restrict the measurement length. We consider only samples that appear in the measurement window:

$w(t) = 1$ if $0 \le t < T$ and $w(t) = 0$ elsewhere. (7)

This rectangular window, together with its spectrum (the phase is omitted), is shown in Figure 4

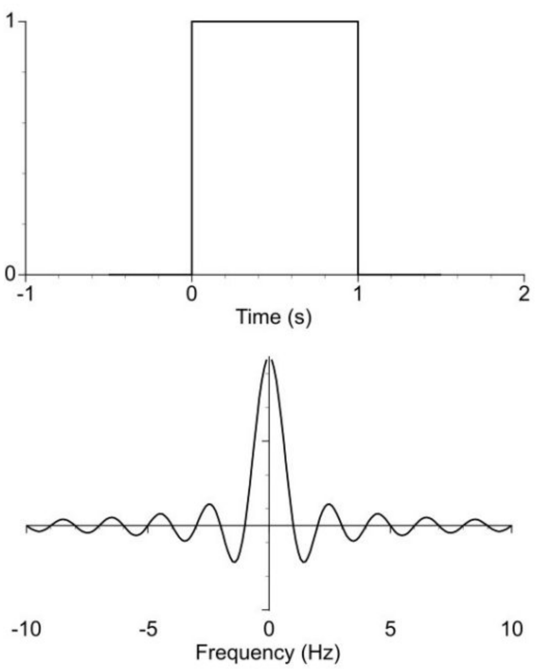

Figure 4. Rectangle window, and its spectrum (the phase is omitted).

This window is called a rectangular window and its major characteristic is its width T. Its spectrum $W(j2\pi f)$ is a sinc-like ($\sin(x)/x$) signal, see Eq. (8), with zero crossings at the multiples of $1/T$. In this example $T = 1\,\text{s}$. This window is multiplied with the sampled signal to obtain a new signal that is only different from zero in a finite number of samples.

The spectra have to be convoluted in the frequency domain. Remembering that a convolution with a Dirac impulse is nothing other than a shift of the origin to the position of the impulse, the result of Figure 5 is found. The broken lines in the spectra indicate the position of the original frequency components. As can be seen, the restriction of the signal to a finite interval in the time domain smears the power in the frequency domain over the neighboring frequencies. This phenomenon is called *leakage*.

Mathematical description:

Time domain: $w(t)\tilde{u}_{\mathrm{d}}(t)$

Frequency domain: $W(j2\pi f) * U_{\mathrm{d}}\left(e^{j2\pi f T_{\mathrm{s}}}\right)$ (8)

With $W(j\omega) = Te^{-j\omega T/2}\operatorname{sinc}(\omega T/2)$ and $\operatorname{sinc}(x) = \sin(x)/x$.

2.2.3. Discretization in Frequency

As can be seen in Figure 5,

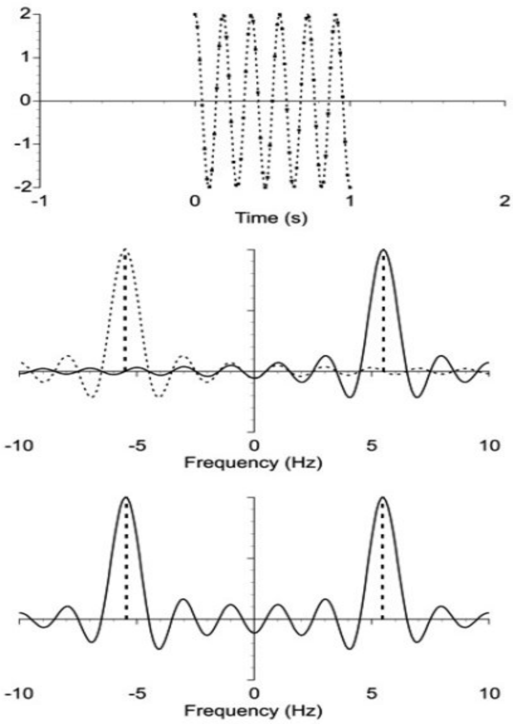

Figure 5. Spectrum of the sampled signal after applying a rectangular window.

The spectrum of the sampled and windowed signal is still a continuous frequency signal. Because the spectrum can only be calculated in a finite number of frequencies, the considered frequencies should also be restricted to a discrete grid. An equidistant grid with spacing $1/T$ is selected. Hence the spectrum is only calculated at the frequencies $f_k = k/T$ Hz. This can be considered as frequency sampling or discretization in frequency. The resulting sampled spectrum shown in Figure 6

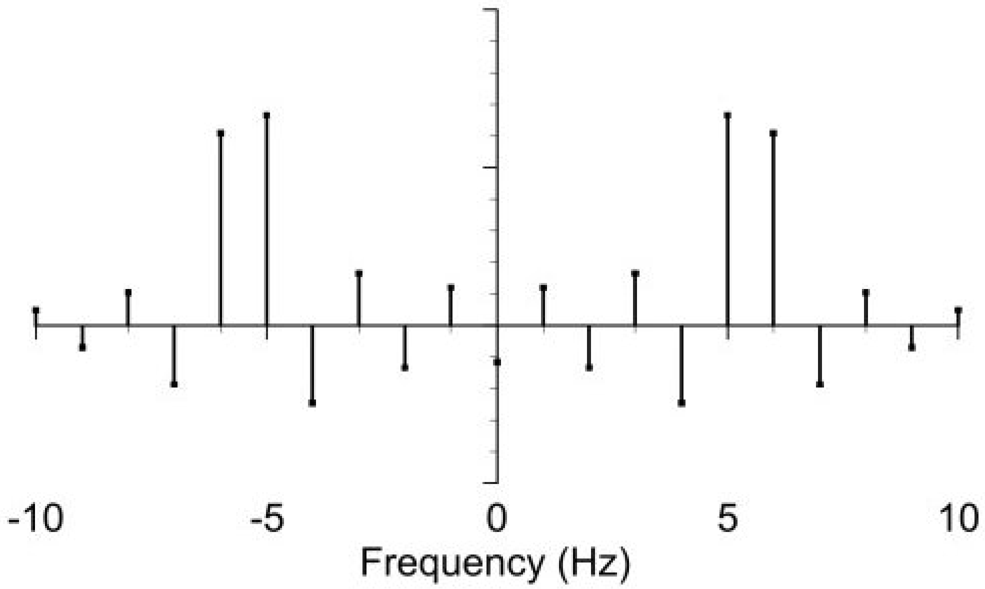

Figure 6. DFT result

is quite disappointing. Although the shape of the original spectrum (Figure 3) can still be recognized, it seems that all detailed information about it has definitely been lost. The basic reason for this problem is that the original frequency (5.5 Hz) does not correspond to one of the sampled frequencies in the DFT (multiples of $1/T = 1\text{Hz}$).

This can also be seen in the time domain representation of the DFT result. Sampling in the frequency domain at multiples of $1/T$ is described as a multiplication with a Dirac train (see Section 2.1) so that in the time domain a convolution should be made with a Dirac train $T\delta_T(t)$. This results in a periodic repetition with period T of the sampled and windowed signal as shown in Figure 7.

However, T is not a multiple of the signal period, resulting in a discontinuity which appears at the borders of the window as seen in Figure 7($T = 1$ s in this case).

Mathematical description:

time domain:

$$\left(w(t)\tilde{u}_{\mathrm{d}}(t)\right) * \left(T\delta_T(t)\right)$$

frequency domain: $\left(W\left(j2\pi f\right)*U_{\mathrm{d}}\left(e^{j2\pi fT_{\mathrm{s}}}\right)\right)\delta_{1/T}\left(f\right)$ (9)

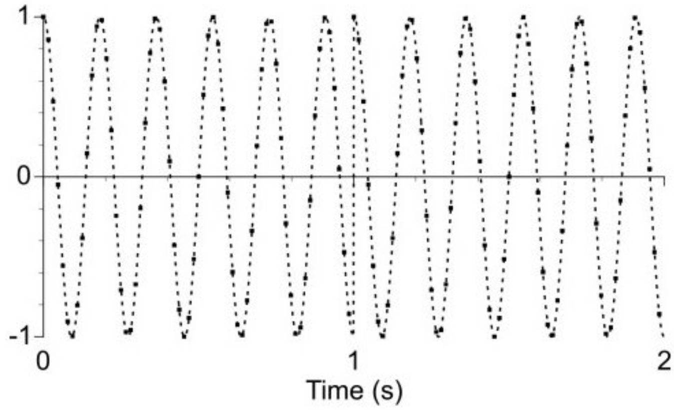

Figure 7. Interpretation of the DFT result in the time domain.

From (9) it follows that the relationship between the time domain samples $u_d\left(n\right)=u\left(nT_{\mathrm{s}}\right)$ (amplitudes of the Dirac impulses of the time domain signal in (9)), and the frequency domain samples $U_{DFT}\left(k\right)$ (amplitudes of the Dirac impulses of the spectrum in (9)), is given by

$$U_{\mathrm{DFT}}\left(k\right)=\sum_{n=0}^{N-1}u\left(nT_s\right)e^{-j2\pi nk/N},k=0,1,\ldots N-1 \quad (10)$$

Eq. (10) is called the discrete Fourier transform (DFT) of the samples $u\left(nT_{\mathrm{s}}\right),n=0,1,\ldots,N-1$.

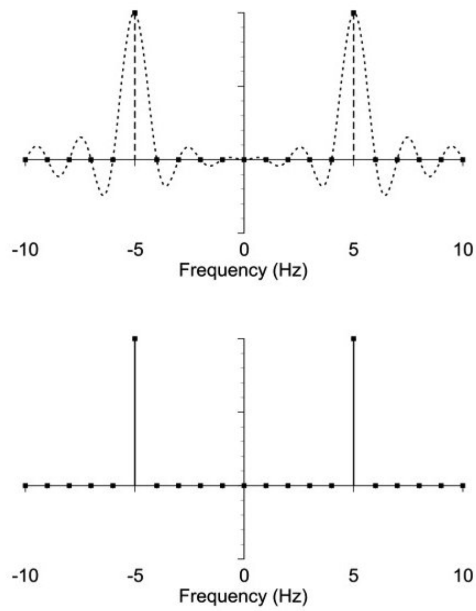

Figure 8. DFT spectrum for a periodic signal when an integer number of periods is measured.

If an integer number of periods is measured, the DFT will give an exact copy of the discrete spectrum of the periodic signal. This is illustrated in Figure 8 showing the spectra after windowing and after discretization for $u(t) = \cos 2\pi f_0 t$, $f_0 = 5$ Hz , $T = 1$ s .

This time no leakage is observed. The basic reason for this remarkable difference is that the continuous-time spectrum equals zero at those frequencies where the spectrum is sampled because the window length is an exact multiple of the period length. Also the time domain interpretation in Figure 9 illustrates the result: this time the periodic repetition coincides with the period of the signal (no discontinuities appear at the multiples of *T*)

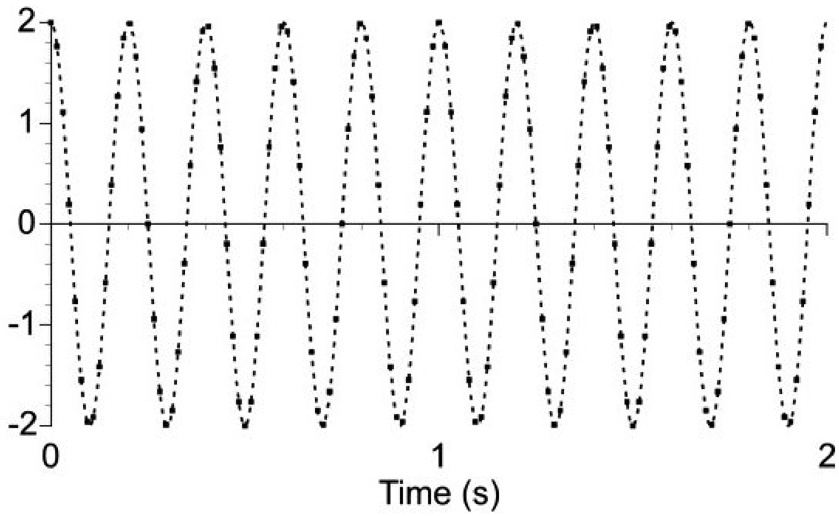

Figure 9. Interpretation of the DFT result in the time domain when an integer number of periods is measured.

At a glance, this seems to be a theoretical result without practical value. The probability of getting an exact match between the signal and the window length is in general, indeed, zero. However, in many FRF measurements, the user masters the generator and the acquisition. In these experimental setups both systems are driven by mother clocks that are synchronized to each other.

It is therefore possible for the user to create this ideal match which eliminates the leakage effect completely. We strongly advise realization of such a setup whenever possible.

If for some reason it is impossible to get synchronized measurements, there exist other less attractive alternatives based on windows other than the rectangular window. An extended discussion of the window properties can be found in the literature. In Section 2.3.we will briefly touch on this topic.

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Biographical Sketches

Johan Schoukens was born in Belgium in 1957. He received the degree of engineer in 1980 and the degree of doctor in applied sciences in 1985, both from the Vrije Universiteit Brussel, Brussels, Belgium. He is presently professor at the Vrije Universiteit Brussel. The prime factors of his interest are in the field of system identification for linear and nonlinear systems, and growing tomatoes in his green house.

Rik Pintelon was born in Gent, Belgium, on December 4, 1959. He received the degree of electrical engineer (burgerlijk ingenieur) in July 1982, the degree of doctor in applied sciences in January 1988, and the qualification to teach at university level (geaggregeerde voor het hoger onderwijs) in April 1994, all from the Vrije Universiteit Brussel (VUB), Brussels, Belgium. From October 1982 till September 2000 he was a researcher of the Fund for Scientific Research - Flanders at the VUB. Since October 2000 he is professor at the VUB in the Electrical Measurement Department (ELEC). His main research interests are in the field of parameter estimation / system identification, and signal processing.