CONTINUOUS-TIME IDENTIFICATION

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Contents

Summary

This chapter considers the problem of estimation of the transfer function of a continuous-time dynamic system in the presence of colored noise. Whereas parameter estimation can be made by means of a discrete-time maximum-likelihood algorithm, an operator transformation permits a continuous-time model parameterization. The method is useful in cases where it is important to estimate the coefficients of a continuous-time transfer function and to maintain a physical interpretation of the transfer function results.

1. Introduction

An accurate knowledge of a continuous-time transfer function is a prerequisite to many methods in physical modeling and control system design. System identification, however, is often made by means of time-series analysis applied to discrete-time transfer function models. As yet there is no undisputed algorithm for parameter translation from discrete-time parameters to a continuous-time description. Problems in this context are associated with the transformation of the system zeros from the discrete-time model to the continuous-time model whereas the system poles are mapped by means of complex exponentials. As a result, a poor parameter transformation tends to affect both the frequency response such as the Bode diagram and the transient response such as the impulse response. One source of error in many existing algorithms is that computation of the system zeros is affected by the assumed and actual inter-sample behavior of the control variables.

There are two circumstances that favor the traditional indirect approach via discrete-time identification: Firstly, data are in general available as discrete measurements. Another problem is the mathematical difficulty to treat continuous-time random processes. In the context of discrete-time measurements, however, it is in many cases sufficient to model disturbances as a noise sequence of finite spectral range.

A relevant question is, of course, why there is no analogue to ARMAX models for continuous-time systems. One reason is that polynomials in the differential operator can not be used for identification immediately due to the implementation problems associated with differentiation. The successful ARMAX-models correspond to transfer function polynomials in the z-transform variable z or z^{-1}—*i.e.*, the forward or the backward shift operators, with advantages for modeling and signal processing, respectively, and translation between these two representations is not difficult. A related problem is how to identify accurate continuous-time transfer functions from data and, in particular, how to obtain good estimates of the zeros of a continuous-time transfer function. The difficulties to convert a discrete-time transfer function to continuous-time transfer function are well known and related to the mapping $f(z) = (\log z)/h$.

We derive an algorithm that fits continuous-time transfer function models to discrete-time data and we adopt a hybrid modeling approach by means of a discrete-time disturbance model and a continuous-time transfer function.

2. A model Transformation

This algorithm introduces an algebraic reformulation of transfer function models. In addition, we introduce discrete-time noise models in order to model disturbances. The idea is to find a causal, stable, realizable linear operator that may replace the differential operator while keeping an exact transfer function. This shall be done in such a way that we obtain a linear model for estimation of the original transfer function parameters a_i, b_i. We will consider cases where we obtain a linear model in all-pass or low-pass filter operators. Actually, there is always a linear one-to-one transformation which relates the continuous-time parameters and the convergence points for each choice of filter.

Then follow investigations on the state space properties of the introduced filters and the original model. The convergence rate of the parameter estimates is then considered. Finally, there are two examples with applications to time-invariant and time-varying systems, respectively. Consider a linear n-th order transfer operator formulated with a differential operator $p = d/dt$ and unknown coefficients a_i, b_i.

$$G_0(p) = \frac{b_1 p^{n-1} + \cdots + b_n}{p^n + a_1 p^{n-1} + \cdots + a_n} = \frac{B(p)}{A(p)}, \qquad (1)$$

where it is assumed that A and B are coprime. It is supposed that the usual isomorphism between transfer operators and transfer functions, *i.e.* the corresponding functions of a complex variable *s*, is valid. Because of this isomorphism, G_0 will sometimes be regarded as a transfer function and sometimes as a transfer operator. A notational difference will be made with *p* denoting the differential operator and *s* denoting the complex frequency variable of the Laplace transform.

It is a necessary requirement on any transfer function that describes a physically realizable continuous-time system because pure derivatives of the input cannot be implemented. This requirement is fulfilled as $\lim_{s\to\infty} G_0(s)$ is finite, *i.e.*, $G_0(s)$ has no poles at infinity. An algebraic approach to system analysis may be suggested. Let *a* be point on the positive real axis and define the mapping

$$f(s) = \frac{a}{s+a}, \qquad s \in \mathbb{C}$$

Let $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ be the complex plane extended with the 'infinity point'. Then *f* is a bijective mapping from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ and it maps the 'infinity point' to the origin and $-a$ to the 'infinity point'. The unstable region—*i.e.*, the right half plane ($\operatorname{Re} s > 0$)—is mapped onto a region which does not contain the 'infinity point'. Introduction of the operator

$$\lambda = f(p) = \frac{a}{p+a} = \frac{1}{1+p\tau}, \qquad \tau = 1/a . \qquad (2)$$

This allows us to make the following transformation

$$G_0(p) = \frac{B(p)}{A(p)} = \frac{B^*(\lambda)}{A^*(\lambda)} = G_0^*(\lambda)$$

with

$$A^*(\lambda) = 1 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots + \alpha_n\lambda^n \qquad (3)$$

$$B^*(\lambda) = \beta_1\lambda + \beta_2\lambda^2 + \cdots + \beta_n\lambda^n . \qquad (4)$$

An input-output model is easily formulated as

$$A^*(\lambda)y(t) = B^*(\lambda)u(t)$$

or on regression form

$$y(t) = -\alpha_1[\lambda y](t) - \cdots - \alpha_n[\lambda^n y](t) + \beta_1[\lambda u](t) + \cdots + \beta_n[\lambda^n u](t) . \qquad (5)$$

This is now a linear model of a dynamical system at all points of time. Notice that $[\lambda u],[\lambda y]$ etc. denote filtered inputs and outputs. The parameters α_i, β_i may now be estimated by any suitable method for estimation of parameters of a linear model. A reformulation of the model (5) to a linear regression form is

$$y(t) = \varphi_\tau^T(t)\theta_\tau, \quad \theta_\tau = \begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n & \beta_1 & \dots & \beta_n\end{pmatrix}^T, \tag{6}$$

$$\varphi_\tau(t) = \begin{pmatrix}-[\lambda y](t), & -[\lambda^2 y](t), & \dots, & [\lambda u](t), & \dots & [\lambda^n u](t)\end{pmatrix}^T. \tag{7}$$

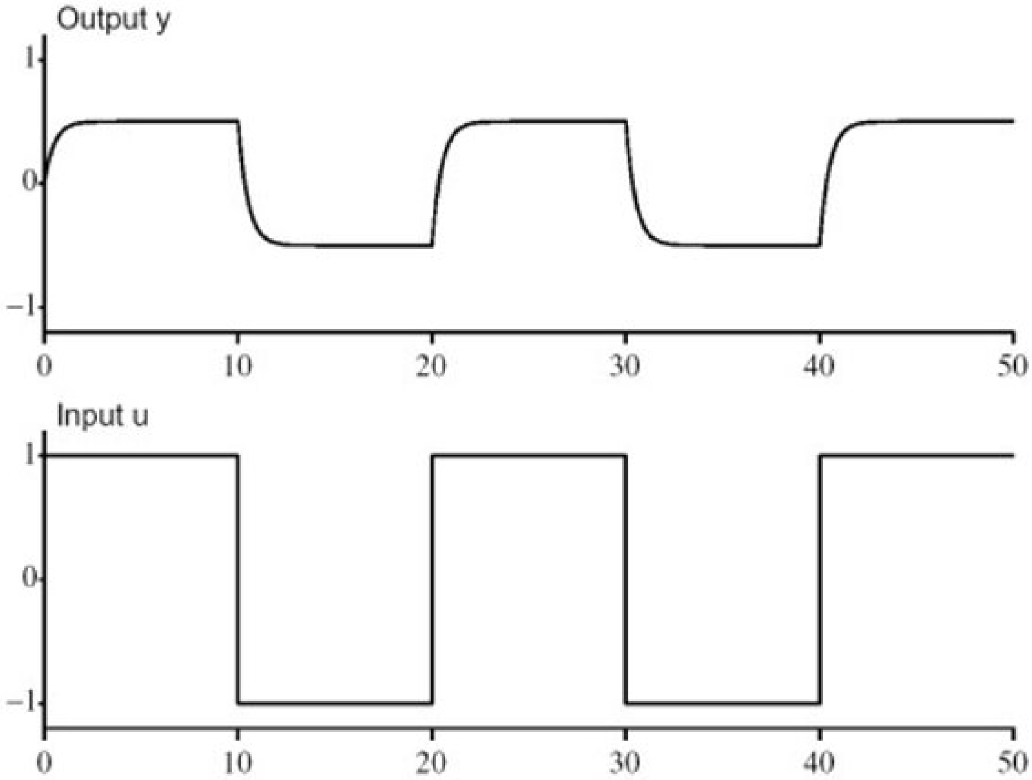

Figure 1: Input *u* and output *y*

with parameter vector θ_τ and the regression vector φ_τ. We may now have the following continuous-time input-output relations:

$$\begin{aligned}
&y(t) = G_0(p)u(t) = G_0^*(\lambda)u(t), \quad Y(s) = \mathcal{L}\{y(t)\} = G_0^*(\lambda(s))U(s)\\
&y(t) = \varphi_\tau^T(t)\theta_\tau\\
&Y(s) = \Phi_\tau^T(s)\theta_\tau \quad \text{where} \quad \Phi_\tau(s) = \mathcal{L}\{\varphi_\tau(t)\}(s),
\end{aligned} \tag{8}$$

where $\mathcal{L}$ denotes a Laplace transform. As a consequence of the linearity of the Laplace transform, one can conclude that the same linear relation holds in both the time domain and the frequency domain. Notice that this property holds without any approximation or any selection of data.

Example—Estimation of two constant parameters

Consider the system with input *u*, output *y*, and the transfer operator G_0

$$y(t) = G_0(p)u(t) = \frac{b_1}{p + a_1}u(t)$$

Use the operator transformation λ of (2)

$$\lambda = \frac{1}{1 + p\tau}, \tag{9}$$

This gives the transformed model

$$G_0^*(\lambda) = \frac{b_1\tau\lambda}{1 + (a_1\tau - 1)\lambda} = \frac{\beta_1\lambda}{1 + \alpha_1\lambda}$$

A linear estimation model of the type (6) is given by

$$y(t) = -\alpha_1[\lambda y](t) + \beta_1[\lambda u](t) = \varphi_\tau^T(t)\theta_\tau(t) \tag{10}$$

with regressor $\varphi_\tau(t)$ and the parameter vector θ_τ and

$$\varphi_\tau(t) = \begin{pmatrix} -[\lambda y](t) \\ [\lambda u](t) \end{pmatrix}, \quad \theta_\tau = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \tag{11}$$

The original parameters are found via the relations

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\tau}(\alpha_1 + 1) \\ \frac{1}{\tau}\beta_1 \end{pmatrix} \tag{12}$$

and their estimates from

$$\begin{pmatrix} \hat{\alpha}_1 \\ \hat{b}_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\tau}(\hat{\alpha}_1 + 1) \\ \frac{1}{\tau}(\hat{\beta}_1) \end{pmatrix} \tag{13}$$

Sampling of all variables in Eq. (10) and application of the recursive least-squares estimation algorithm is obviously possible. Simulation results for different choices of the filter time constant τ [s] and the sampling interval $h = 0.03$ [s] are based on the input-output data of Fig. 1. All simulations have started with initial values at zero for the parameter estimates and the filters. The simulations have been performed with $a_1 = 2$ and $b_1 = 1$ and with a square wave as a moderately exciting input, see Fig. 2.

The simulations in Fig. 2 indicate that the convergence works properly with accurate estimates over at least two decades of values of τ. The convergence rate is faster for a shorter τ but the convergence transient may be violent for 'too' short time constants τ. Figures 1 and 2 demonstrate convergence rates over a large range of values of the time constant τ and that the convergence rate is higher for small values of τ. The sampling rate seems not to be a limiting factor for the convergence rate.

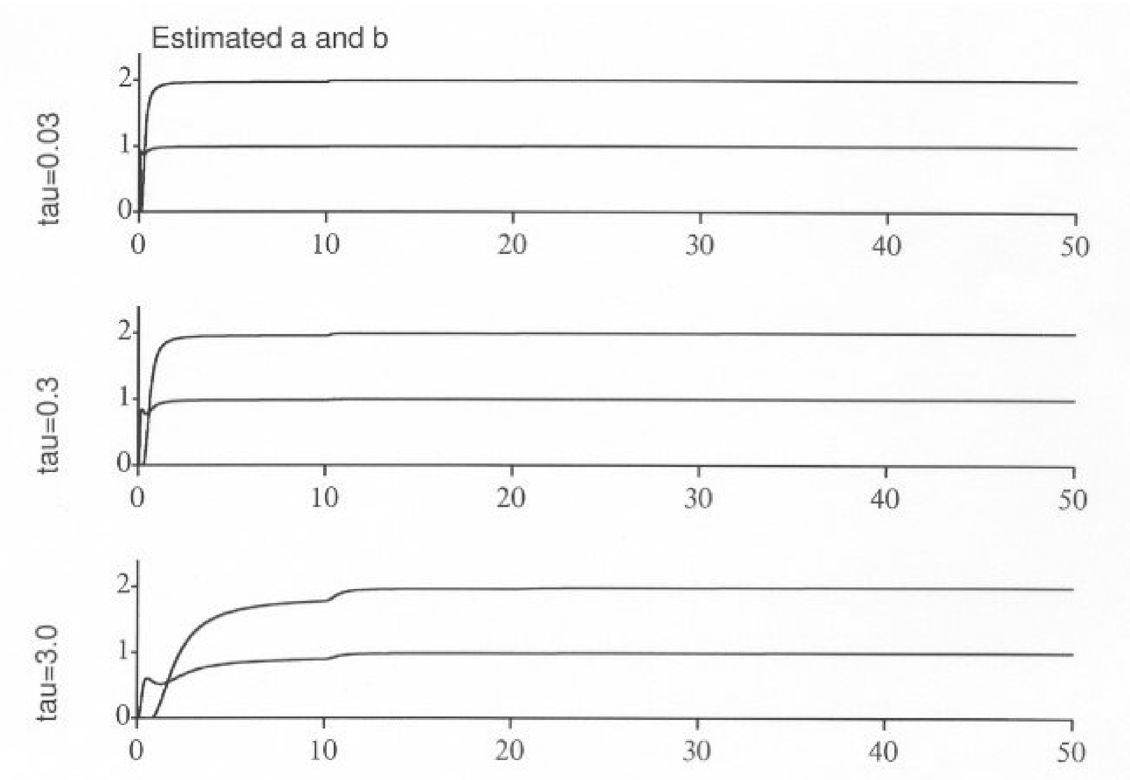

Figure 2: Estimates $\hat{a}_1$ and $\hat{b}_1$

The filter constant a (or τ) of the operator λ should therefore be regarded as a design parameter to be chosen appropriately. As the components of the regression vector φ_τ tend to become small for high frequency input one should match the filter constant with respect to the dynamics of the system investigated.

Remark—Operator Representation Singularities

A relevant question is, of course, how general is the choice λ and if it can, for instance, be replaced by some other bijective mapping

$$\mu = \frac{bs+a}{s+a}, \qquad b \in \mathbb{R}, \quad a \in \mathbb{R}^+, \qquad \text{and } s = \frac{\mu a - a}{b - \mu} \tag{14}$$

One can treat this problem by considering the example

$G_0(s) = \dfrac{1}{s + a/b + \epsilon}$ where $\epsilon \in \mathbb{R}$ is small

Application of the operator translation μ gives

$$G_0(s) = \frac{1}{s + a/b + \epsilon} = \frac{\mu - b}{-\epsilon b + (a(\frac{1}{b} - 1) + \epsilon)\mu} = G_0^*(\mu)$$

Obviously, the zero-order denominator polynomial coefficient will vanish for $\epsilon = 0$ so that $G_0^*(\mu)$ exhibits a pole at $z = 0$. The corresponding estimation model would be

$$y = \alpha[\mu y] + \beta_1[\mu u] + \beta_0[u] = (\frac{1}{\epsilon}\frac{a}{b}(\frac{1}{b} - 1) + \frac{1}{b})[\mu y] - \frac{1}{\epsilon b}[\mu u] + \frac{1}{\epsilon}[u]$$

which exhibits coefficients of very large magnitudes for small ϵ. This would constitute a serious sensitivity problem — at least for $b > 0$ for which $G_0(s)$ is stable. An operator μ with $b < 0$ according to Eq. (14) would give rise to large coefficients of the transformed model only for unstable systems which might be more 'affordable'. By comparison, a model transformation using λ would not exhibit any such singularities.

Hence, use of the operator μ should for sensitivity reasons be restricted to cases with $b = 0$ (or $b_{min} < b \le 0$ for some number b_{min} chosen according to some *a priori* information about the system dynamics). Note that the set of polynomials associated with $b < 0$ is related to the orthogonal Laguerre polynomials.

2.1. Parameter Transformations

Before we proceed to clear cut signal processing aspects we should make clear the relationship between the parameters α_i, β_i of (5) and the original parameters a_i, b_i of the transfer function (1). Let the vector of original parameters be denoted by

$$\theta = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_n & b_1 & \dots & b_n \end{pmatrix}^T \tag{15}$$

Using the definition of λ (2) and (5) it is straightforward to show that the relationship between (6) and (19) is

$$\theta_\tau = F_\tau \theta + G_\tau \tag{16}$$

where the $2n \times 2n-$matrix F_τ is

$$F_\tau = \begin{pmatrix} M_\tau & 0_{n\times n} \\ 0_{n\times n} & M_\tau \end{pmatrix} \tag{17}$$

and where

$$M_\tau = \begin{pmatrix} m_{11} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} & \end{pmatrix}, \quad m_{ij} = (-1)^{i-j}\binom{n-j}{i-j}\tau^j \tag{18}$$

Furthermore, the $2n \times 1-$vector G_τ are given by

$$G_\tau = \begin{pmatrix} g_1 & \dots & g_n & 0 & \dots & 0 \end{pmatrix}^T; \qquad g_i = \binom{n}{i}(-1)^i \tag{19}$$

The matrix F_τ is invertible when M_τ is invertible, *i.e.* for all $\tau > 0$. The parameter transformation is then one-to-one and

$$\theta = F_\tau^{-1}(\theta_\tau - G_\tau)$$

We may then conclude that the parameters a_i, b_i of the continuous-time transfer function G_0 may be reconstructed from the parameters α_i, β_i of θ_τ by means of basic matrix calculations. As an alternative we may estimate the original parameters a_i, b_i of θ from the linear relation

$$y(t) = \theta_\tau^T \varphi_\tau(t) = (F_\tau\theta + G_\tau)^T \varphi_\tau(t) \tag{20}$$

where F_τ and G_τ are known matrices for each τ. Furthermore, elaborated identification algorithms adapted for numerical purposes sometimes contain some weighting or orthogonal linear combination of the regression vector components by means of some linear transformation matrix *T*. Thus, one can modify (24) to

$$y(t) = (T\varphi_\tau(t))^T T^{-T} F_\tau \theta + (T\varphi_\tau(t))^T T^{-T} G_\tau$$

Hence, the parameter vectors θ_τ and θ are related via known and simple linear relationships so that translation between the two parameter vectors can be made without any problem arising. Moreover, identification can be made with respect to either θ or θ_τ.

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Biographical Sketch

Rolf Johansson received the Master-of-Science degree in Technical Physics in 1977, the Bachelor-of-Medicine degree in 1980, the doctorate in control theory 1983, was appointed Docent in 1985, and received the Doctor-of-Medicine degree (M.D.) in 1986, all from Lund University, Lund, Scandinavia. He is member of SIAM and IEEE and is a Fellow of the Swedish Society of Medicine. Currently, he is Professor of control theory at the Lund Institute of Technology. In his scientific work, he has been involved in research in adaptive system theory, mathematical modeling, system identification, robotics and signal processing. Since 1987, he has also participated in research and as a graduate advisor at the Faculty of Medicine, Lund University Hospital.

He has had the following visiting appointments: Researcher, 1985, Centre National de la Recherche Scientifique (CNRS), Grenoble, France; Visiting scientist CalTech, CA, June 1997, June 2001; Rice

Univ., Houston, TX, May 1998, May 2001; Supélec, Paris, France, June 1998; Univ. Illinois at Urbana-Champaign, IL; UC Santa Barbara, CA, Aug. 1999; Univ. Napoli Fed II, Italy, July 2000; Guest Professor, NTNU, Trondheim, NO, Aug. 2001.

Rolf Johansson was awarded the 1995 biomedical engineering prize (the Ebeling Prize) of the Swedish Society of Medicine for distinguished contribution to the study of human balance through application and development of system analysis and robotics.

In 1993 he published the book System Modeling and Identification, Prentice Hall, Englewood Cliffs, NJ (Information and System Sciences Series Ed. T. Kailath). Currently he is coordinating director in robotics research with participants from several departments of Lund University.