# NONLINEAR-MODEL CASE

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**Keywords:** Bound-based identification, set-membership identification, parameter-bounding, unknown-but-bounded uncertainty, bounded noise, nonlinear regression models, nonlinear state-space models, parameter estimation, algorithms.

## Contents



## Summary

In this paper, an overview of non-linear parameter bounding approaches, to solve a general parameter estimation problem under unknown-but-bounded uncertainty, is presented. The proposed algorithms are classified according to their leading principle, such as (i) intersection of feasible parameter sets determined by the measurement uncertainty bounds, (ii) encapsulation of the feasible parameter set by e.g., supporting hyperplanes, (iii) discrete approximation of the feasible parameter set by point estimates or a collection of e.g., boxes, ellipsoids, etc. (iv) projection of the measurement uncertainty set into the parameter space, and (v) development of algorithms for special model classes using specific information. The approaches will be illustrated by a simple exponential model with two parameters.



## 1. Introduction

During the last two decades a growing amount of literature on so-called set-membership identification or parameter-bounding approaches has become available (see *Bound-based Identification*). The key problem in this bound-based identification is not to find a single vector with optimal parameter estimates, but a *set* of feasible parameter vectors that are consistent with a given model structure and data with bounded uncertainty. A bounded error characterization, as opposed to a statistical characterization in terms of mean, variances or probability distributions, is favored when the central limit theorem is inapplicable, such as in situations with small data sets or with heavily structured (modeling) errors. If the error is $\ell_\infty$ bounded and the model is linear in the parameters this feasible set is a polytope, which complexity depends on the number of data and especially on the parameter dimensionality. See *Linear-model Case* for a further treatment of linear parameter bounding approaches.

However, in addition to the linear case, in practice, many non-linear parameter estimation problems are encountered. In this chapter, solutions to this problem given bounded noise data are investigated and illuminated.

In section 2, some definitions are given. On the basis of these definitions the estimation problem is formulated. The approaches to solve the non-linear bounded parameter estimation problem are presented in section 3. Then, in section 4, an illustrative example is presented and the results are discussed. Finally, in section 5 some concluding remarks are given.

## 2. Definitions and notation

Consider the following non-linear regression type of model

$$\mathbf{y} = \mathbf{F}(\vartheta) + \mathbf{e} \tag{1}$$

where $\mathbf{y} \in \mathbb{R}^N$ contains the observed output data, $\mathbf{F}(\vartheta)$ is a non-linear vector function mapping the unknown parameter vector $\vartheta \in \mathbb{R}^m$ into a noise-free model output $\hat{\mathbf{y}}$. The information uncertainty vector $\mathbf{e}$ is assumed to be bounded in a given norm. In what follows, it is assumed that it is bounded in the $\ell_\infty$ norm, so that

$$\|\mathbf{e}\|_\infty \le \varepsilon \tag{2}$$

where $\varepsilon$ is a fixed positive number, and leading to the error set

$$\Omega_e := \left\{\mathbf{e} \in \mathbb{R}^N : \|\mathbf{e}\|_\infty \le \varepsilon\right\} \tag{3}$$

Hence, a measurement uncertainty set (MUS), containing all possible output measurement vectors consistent with the observed output data and uncertainty characterization, is defined as

$$\Omega_y := \left\{\tilde{\mathbf{y}} \in \mathbb{R}^N : \|\mathbf{y} - \tilde{\mathbf{y}}\|_\infty \le \varepsilon\right\} \tag{4}$$

This set is a hypercube in $\mathbb{R}^N$. Generally speaking, it is an $\ell_\infty$ norm ball in $\mathbb{R}^N$. Let the set

$$\Omega_\vartheta := \left\{\vartheta \in \mathbb{R}^m : \|\mathbf{y} - \mathbf{F}(\vartheta)\|_\infty \le \varepsilon\right\} \tag{5}$$

define the feasible parameter set. Then, the set-membership estimation problem is to characterize this feasible parameter set (FPS), which is consistent with the model (1), the data $(\mathbf{y})$ and uncertainty characterization (2).

For further analysis the image set, which is a *p*-dimensional variety in the *N*-dimensional measurement space, is defined as follows:

$$\Omega_{\tilde{y}} := \left\{ \tilde{\mathbf{y}} \in \mathbb{R}^N : \tilde{\mathbf{y}} = \mathbf{F}(\vartheta);\, \vartheta \in \mathbb{R}^m \right\} \} \qquad (6)$$

The image set related to the FPS, also called the feasible model output set, is then defined as:

$$\Omega_{\hat{y}} := \left\{ \hat{\mathbf{y}} \in \mathbb{R}^N : \hat{\mathbf{y}} = \mathbf{F}(\vartheta);\, \vartheta \in \Omega_\vartheta \right\} = \Omega_{\tilde{y}} \cap \Omega_y \qquad (7)$$

Let us illustrate the introduced sets by a simple example with two measurement and two unknown parameters. Furthermore, the example will also show some of the specific estimation problems in nonlinear bound-based identification.

**Example 1** Suppose the relation $\mathbf{F}(\vartheta)$ is given by: $\mathbf{F}(\vartheta) = \sin(\vartheta_1 t) + \vartheta_2$, and the measurements are: $t(1) = 1;\, y(1) = 1.0$ and $t(2) = 3;\, y(2) = 0.5$ with error bound $\varepsilon = 0.5$. Hence, when only one measurement at $t(1)$ is available $\Omega_y$ is an interval, in this case $[0.5, 1.5]$, and $\Omega_\vartheta$ is an unbounded set that is only bounded by a pair of bounds: $\sin(\vartheta_1) + \vartheta_2 = 0.5$ and $\sin(\vartheta_1) + \vartheta_2 = 1.5$ (see Figure 1). Consequently, the image set is equal to the real axis and the feasible model output set is equal to the measurement uncertainty set.

When the second measurement at $t(2)$ becomes available $\Omega_y$ becomes a square with center $[1\;\; 0.5]^T$ and edges with length one in the measurement space. Consequently, in the parameter space another pair of bounds is added, which together with the bounds related to the first measurement define an *exact* solution to the parameter bounding estimation problem. Notice from Figure 1 that $\Omega_\vartheta$ (dotted regions) becomes a non-connected set with non-convex subsets. Furthermore prior knowledge restricts $\vartheta_1$ to the $[0, 2\pi]$ interval. The image set is equal to $\left\{ \tilde{\mathbf{y}} \in \mathbb{R}^2 : \tilde{\mathbf{y}} = \left[ \vartheta_{11} + \vartheta_2 \;\;\; \vartheta_{12} + \vartheta_2 \right]^T ;\, \vartheta_{11}, \vartheta_{12} \in [-1, 1],\, \vartheta_2 \in \mathbb{R} \right\}$, a strip in $\mathbb{R}^2$, and again the feasible model output set is equal to the MUS (see Figure 2). However, when a third measurement becomes available the feasible model output set will generally not be equal to the MUS and the image set becomes a two-dimensional variety in $\mathbb{R}^3$.

In what follows, the feasible parameter set, or a subset of it, is assumed to be connected albeit non-convex.

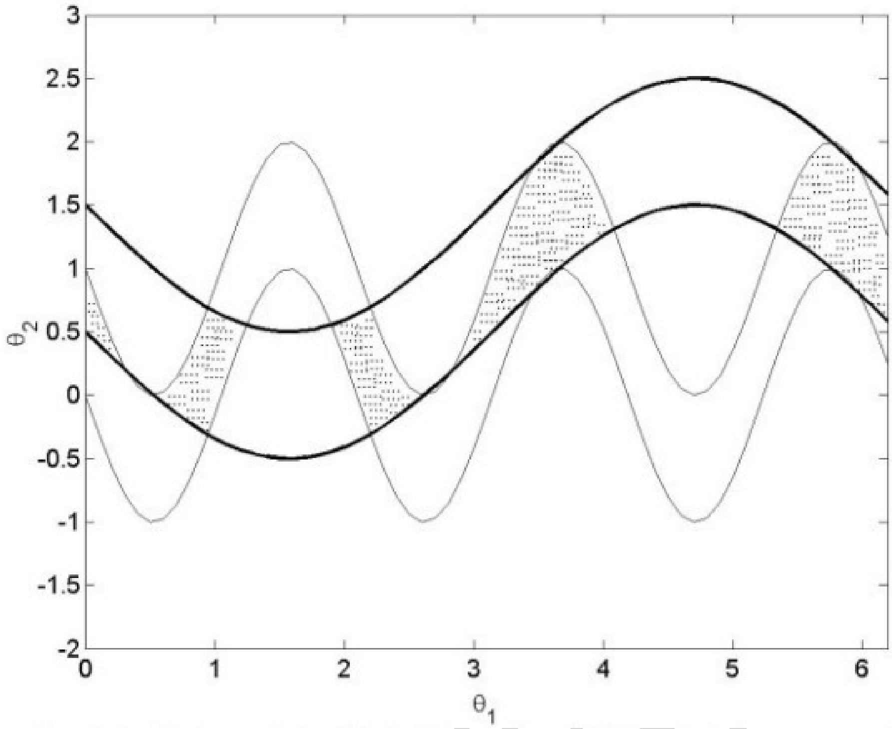


Figure 1. Graphical representation of the feasible parameter set (dotted area) after two measurements.

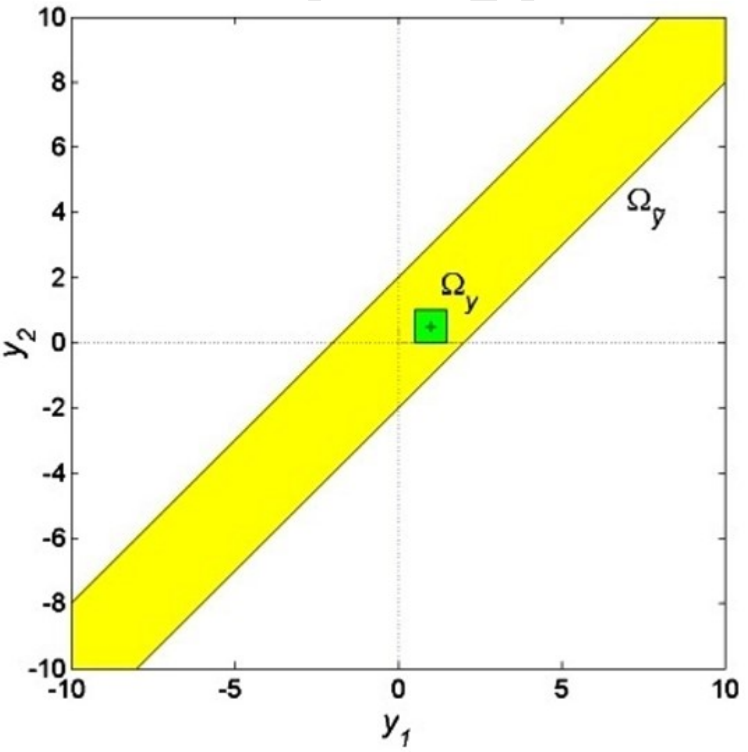


Figure 2. Graphical representation of the MUS ( $=\Omega_{\hat{y}}$ ) and image set ( $\Omega_{\hat{y}}$ ) after two measurements.

## 3. Classification of non-linear parameter bounding algorithms

In order to obtain some insight in the way parameter bounding estimation problems, as defined in the previous section, are solved a classification according to the leading principle is useful and is presented in the following. This classification holds for general models nonlinear in the parameters and for data with point-wise ($\ell_\infty$-norm) bounded noise.

### 3.1. Intersection

As shown in the example, it appears that at sample instant $t$ each measurement with its associated noise bounds defines two bounding surfaces in the parameter space, which bound a feasible parameter region $\left(\Omega_\vartheta(t)\right)$. Hence, each parameter vector situated within this region is consistent with the uncertain measurement. Consequently, the intersection of these individual regions will provide an exact characterization of $\Omega_\vartheta$, that is

$$\Omega_\vartheta := \cap_{t=1}^{N} \Omega_\vartheta(t) \tag{8}$$

These intersection algorithms have never been fully developed due to the large complexity of the problem, except for cases where $m \leq 3$ so that the problem can be solved graphically, as in the previous example.A first attempt to characterize $\Omega_\vartheta$ could be via the determination of vertices or edges. For the determination of a vertex of $\Omega_\vartheta$, for instance, at least $m$ sharp conditions have to be detected. Once these sharp conditions have been found, and this is the most difficult part, the vertex can be found by solving the parameter values from $m$ nonlinear algebraic equations. However, one should realize that in the nonlinear case, when the feasible parameter set is not a polytope, it does not suffice to determine only the vertices nor the edges for a full characterization of the set.

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## Biographical Sketch

**Karel J. Keesman** (born 1956) is Associate Professor in the Systems and Control Group of the Wageningen University. He received his M.Sc. degree in Hydrology and Water Management from the Wageningen University in 1984, and his Ph.D. for his work on set-membership identification and prediction of ill-defined systems, with application to a water quality system, from the University of Twente, The Netherlands, in 1989. From May 1989 to December 1990, he worked as a research fellow on batch process control in the Process Control Group of the Department of Chemical Engineering of the University of Twente. Since 1991, he is with the Systems and Control Group. His research interests include identification, prediction and control of uncertain dynamic systems, in particular environmental and biotechnical systems.