MULTIVARIABLE POLES AND ZEROS

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Contents

Summary

This paper introduces two fundamental concepts for the understanding of linear systems

dynamics and for the analysis and design of control systems, which are those of poles and zeros. The poles of a system are crucial characteristics of the internal system dynamics, and characterize system free response, stability and general aspects of the performance of a system. The poles of a system are affected by the different compensation schemes, such as state, output feedback and dynamic feedback compensation and thus they are the subject of many design methodologies aiming at shaping the internal system dynamics. The current introduction on the characterization of poles and their properties provides the basis for many control designs. The notion of zeros is more complex. Zeros express the interaction between internal dynamics and the effort to control and observe the system and they are thus products of overall system design, that apart from process synthesis involves selection of actuation and measurement schemes for the system. The significance of zeros is mainly due to that they remain invariant under a large set of compensation schemes, in addition to the fact that they define limits of what can be achieved under compensation. This makes zeros crucial for design, since they are part of those factors characterizing the potential of a given system to achieve certain design objectives under compensation. The invariance of zeros implies that their design is an issue that has to be addressed outside the traditional control design; this requires understanding of zero formation process and involves early design stages mechanisms such as process instrumentation. Poles and zeros are conceptually inverse concepts (resonances, anti-resonances) and such mechanisms are highlighted throughout the paper. The emphasis in this article is to provide an overview of the fundamentals of the two notions, examine them from many different perspectives and provide links between them.

1. Introduction

The concepts of pole, and zero have emerged as the key tools of the classical methods of Nyquist-Bode and root locus for the analysis and design of linear, single-input, single –output (SISO) feedback systems. The development of the state space S(A,B,C,D) description, transfer function G(s) description, and complex variable, (g(s), algebraic function) methods for linear multivariable systems has led to a variety of definitions for the zeros and poles in the multivariable case and the emergence of many new properties. The variety and diversity in the definitions for the zeros and poles is largely due to the differences between alternative system representations the difference in approaches used, the objectives and types of problems they have to serve.

Loosely speaking, multivariable poles and zeros are resonant and anti-resonant frequencies respectively, that is to say they are frequencies whose transmission explodes with time, or whose transmission is completely blocked. This, of course, is intuitively appealing since it forms a natural extension of the definitions given for the scalar case, where the poles and zeros of a scalar transfer function are defined as the values of the complex frequency s for which the transfer function gain becomes ∞, or 0 correspondingly. The inversion of roles of poles and zeros suggested by their classical complex analysis definition motivates the dynamic (in terms of trajectories) properties of zeros. The physical problem used to define multivariable zeros is the “output zeroing“ problem, which is the problem of defining appropriate non-zero input exponential signal vectors and initial conditions which result in identically zero output. Such a problem is the dual of the “zero input” problem defining poles, which is the problem of defining appropriate initial conditions, such that with zero input the output is a nonzero exponential vector signal. Those two physical problems emphasize the duality of the roles of poles and zeros.

Apart from its natural dynamic appeal such definitions for poles and zeros have the additional advantages that they reveal the geometric dimension of such concepts as well as their link with fundamental structural invariants of the system. The poles-eigenvalues have a well-defined geometry introduced by the eigenvectors and the corresponding spaces (the A-invariant spaces of the state space). Similarly, the geometry of zeros is linked to generalized eigenvalue-eigenvector problems and corresponding spaces (types of (A, B)- invariant spaces). The Jordan form of the state matrix reveals the invariant structure of poles in the state space set-up. The Smith form, and in some more detail the Kronecker form, of the state space system matrix introduce the zero structure of the state space models; for transfer function models the pole zero structure is introduced by the Smith-McMillan form. Such links reveal the poles as invariants of the alternative system representations under a variety of representation and feedback transformations. The strong invariance of zeros (large set of transformations) makes them critical structural characteristics, which strongly influence the potential of systems to achieve performance improvements under compensation.

The dynamic characterization of zeros leads to algebraic characterizations, which reveal them as by-products of the interaction of the internal dynamics and the model, input, output structure. This is contrary to the pole characterization, which shows that they express the internal dynamics and as these are viewed through the input, output system structure. Such observations lead to that zero design is a task associated with overall selection of inputs, outputs and thus belongs to the earlier system design stage of process instrumentation. In the paper we consider both finite and infinite zeros and examine them in both state space and transfer function context. The relationships between the corresponding notions for state space and transfer function models are revealing interesting links to the fundamental system properties of controllability and observability.

The above classes of zeros are characterized by the property of invariance under appropriate transformations. Not all definitions of zeros, however, possess all the above invariance properties. The algebraic function approach used to generalise Nyquist and Root Locus ideas to the multivariable case introduces an alternative definition of zeros and poles, which do not have all the invariance properties. The relationships between these different classes of zeros are examined and their significance to control problems is briefly discussed.

Every square system (same number of input and outputs) has zeros (finite and/or infinite); however, non-square systems generally do not have zeros and this is an important difference with the poles that exist independent from input, output dimensionalities. Closing feedback loops creates square systems and this involves creation of zeros; such phenomena are within the area of designing zeros and the mechanisms for zero formation are examined. Although non-square systems generically have no zeros, they have “almost zeros”; this extended notion is also introduced, express “almost pole-zero cancellations” and it is shown that in a number of cases behaves like the exact notion. The significance of zeros for control problems is finally discussed.

Poles and zeros are fundamental system concepts with dynamic, algebraic, geometric, feedback and computational aspects. The paper provides an overview of the key aspects and a detailed account of the material that may be found in the cited references.

2. System Representations and their Classification

Linear time invariant multivariable systems are represented in the time domain by state variable model

$$\dot{x} = Ax + \mathbf{B}u$$

$$\mathbf{S}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) \qquad (1)$$

$$y = Cx + Du$$

where **x** is an n-vector of the state variables, **u** is an p-vector of inputs and **y** is an m-vector outputs. **A**, **B**, **C**, **D** are respectively nxn, nxp, mxn, mxp matrices. The above description may be represented in an autonomous or implicit form as:

$$\mathbf{S}(\boldsymbol{\Phi}, \boldsymbol{\Omega}): \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix}}_{\triangleq \boldsymbol{\Phi}} \underbrace{\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{u}} \\ \dot{\mathbf{y}} \end{bmatrix}}_{\triangleq \dot{\boldsymbol{\xi}}} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{O} \\ \mathbf{C} & \mathbf{D} & -\mathbf{I} \end{bmatrix}}_{\triangleq \boldsymbol{\Omega}} \underbrace{\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \\ \mathbf{y} \end{bmatrix}}_{\triangleq \boldsymbol{\xi}} \qquad (2)$$

where $\boldsymbol{\Phi}$, $\boldsymbol{\Omega}$ are the coefficient matrices and $\boldsymbol{\xi} = [\mathbf{x}^t, \mathbf{u}^t, \mathbf{y}^t]^t$ is the composite vector, or implicit vector of the state space description. The vector $\boldsymbol{\xi}$ contains the state, input and output vectors and makes no distinction between them. $\mathbf{S}(\boldsymbol{\Phi}, \boldsymbol{\Omega})$ description belongs to the general class of generalized autonomous differential descriptions.

$$\mathbf{S}(\mathbf{F},\mathbf{G}): \; F\dot{z} = Gz \qquad (3)$$

where **F**, **G** are rxk matrices and **z** is a k-vector. The above system is characterized by the matrix pencil p**F**-**G**, where p $\triangleq d/dt$ denotes the derivative operator; p**F**-**G** completely characterize the state space description and it is referred to as *implicit system pencil*.

An alternative matrix pencil form for the state space description is obtained by taking Laplace transforms of (1) which lead to the s-domain description.

$$s\,\tilde{x}(s) - x(0) = A\tilde{x}(s) + B\tilde{u}(s)$$
$$\tilde{y}(s) = C\,\tilde{x}(s) + D\tilde{u}(s)$$

where $\tilde{x}(s)$, $\tilde{u}(s)$, $\tilde{y}(s)$ denote the Laplace transforms of **x**(t), **u**(t), **y**(t) vectors respectively and **x**(0) the initial value of **x**(t). We may express the above in a matrix form as

$$\begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} \tilde{x}(s) \\ \tilde{u}(s) \end{bmatrix} = \begin{bmatrix} x(0) \\ -\tilde{y}(s) \end{bmatrix},$$
$$P(s) = \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \tag{4}$$

and the matrix coefficient **P**(s) is a matrix pencil entirely characterizing the state space model and it is known as the Rosenbrock *System Matrix Pencil.* The input output, or transfer function model is described by

$$\tilde{y}(s) = \mathbf{G}(s)\,\tilde{u}(s) \tag{5}$$

where **G**(s) is an mxp rational matrix. The transfer function may also be described in a matrix fraction description form as

$$\mathbf{G}(s) = \mathbf{N}_r(s)\,\mathbf{D}_r(s)^{-1} = \mathbf{D}_\ell(s)^{-1}\,\mathbf{N}_\ell(s) \tag{6}$$

where $\mathbf{N}_r(s)$, $\mathbf{N}_\ell(s)$ are the mxp right, left polynomial matrix numerators respectively and $\mathbf{D}_r(s)$, $\mathbf{D}_\ell(s)$ are the pxp, mxm polynomial matrix denominators correspondingly. It will be assumed that $\mathbf{D}_r(s)$, $\mathbf{N}_r(s)$ are right coprime and $\mathbf{D}_\ell(s)$, $\mathbf{N}_\ell(s)$ are left coprime (see polynomial and Matrix Fraction descriptions). Using (5) and the factorization (6) we can readily obtain the following description

$$\mathbf{D}_\ell(s)\,\tilde{y}(s) = \mathbf{N}_\ell(s)\,\tilde{u}(s) \tag{7}$$

and by introducing the vector $\tilde{h}(s) = \mathbf{D}_\ell(s)^{-1}\,\tilde{u}(s)$ the description

$$\tilde{y}(s) = \mathbf{N}_r(s)\tilde{h}(s),\ \ \tilde{u}(s) = \mathbf{D}_r(s)\tilde{h}(s) \tag{8}$$

The above two lead to the following input output type representations for the system

$$[D_\ell(s), N_\ell(s)]\begin{bmatrix} \tilde{y}(s) \\ -\tilde{u}(s) \end{bmatrix} = 0,$$
$$T_\ell(s) = \left[D_\ell(s), N_\ell(s)\right] \tag{9}$$

$$\begin{bmatrix} \tilde{y}(s) \\ \tilde{u}(s) \end{bmatrix} = \begin{bmatrix} N_r(s) \\ D_r(s) \end{bmatrix} \tilde{h}(s), \quad T_r(s) = \begin{bmatrix} N_r(s) \\ D_r(s) \end{bmatrix} \tag{10}$$

The first, based on $\mathbf{T}_{\ell}(s)$, is referred to *as* Kernel input-output description, whereas the second based on $\mathbf{T}_r(s)$ as a parametric input-output description. The matrices $\mathbf{T}_{\ell}(s)$, $\mathbf{T}_r(s)$ based on coprime MFDs will be called left-, right- composite matrices.

3. Background on Polynomial Matrices and Matrix Pencils

The study of zero structure of linear systems represented by state space, or transfer function models heavily relies on the theory of polynomial matrices (see *Polynomial and Matrix Fraction Description*) and matrix pencils (see *Canonical Forms of State Space Models*); here we review some of the fundamentals of their structure and introduce some useful notation. We consider matrices **T**(s) of dimension qxr with elements from the field of rational functions ***R***(s), or the ring of polynomials ***R***[s]; such matrices are called respectively rational, polynomial. The rank of **T**(s) over ***R***(s) is denoted by ρ = rank {**T**(s)} and will be called the normal rank of **T**(s). **T**(s) may be viewed as a function of the complex variable s and thus for some s = z, rank {**T**(s)} = ρ_z < ρ; such values s =z are called zeros of **T**(s) and ρ_z is called the local rank of **T**(s). The structure of zeros of **T**(s) is linked to study of certain form of equivalence defined on such matrices, which reveals the zeros as roots of invariant polynomials.

Let $\mathbf{T}_1$(s), $\mathbf{T}_2$(s) be q x r polynomial matrices. These matrices are called ***R***[*s*]-uni-modular equivalent, or simply ***R***[s]-equivalent, if there exist polynomial matrices $\mathbf{U}_{\ell}$(s) $\mathbf{U}_r$(s) of dimension qxq, rxr respectively with the property $|\mathbf{U}_r(s)| = c_1 \neq 0$, $|\mathbf{U}_{\ell}(s)| = c_2 \neq 0$ ($|\cdot|$ denotes determinant) and called ***R***[*s*]-uni-modular, such that:

$$\mathbf{T}_1(s) = \mathbf{U}_{\ell}(s)\, \mathbf{T}_2(s)\, \mathbf{U}_r(s) \tag{11}$$

The above relation introduces an equivalence and for any matrix **T**(s) we have an equivalence class and associated invariants which are described by the following result.

Smith Form Theorem: If **T**(s) is a q x r polynomial matrix with normal rank $\rho \leq$ min (q, r) there exist uni-modular matrices $\mathbf{U}_{\ell}$(s), $\mathbf{U}_r$(s) such that

$$\mathbf{U}_{\ell}(s)\, \mathbf{T}(s)\, \mathbf{U}_r(s) = \left[\begin{array}{cccc|c} f_1(s) & & & & O \\ & \ddots & & & \\ & & \ddots & & \\ & & & f_\rho(s) & \\ \hline & O & & & O \end{array}\right] = \mathbf{S}(s) \tag{12}$$

where **S**(s) is qxr polynomial matrix $f_1(s),\ldots, f_\rho(s)$ are uniquely defined and $f_1(s)/f_2(s)/\ldots/f_\rho(s)$ denotes the successive divisibility (i.e., $f_1(s)$ divides $f_2(s)$ etc).

The polynomials $f_i(s)$ are called invariant polynomials of **T**(s) and the set $\{f_i(s), i=1,\ldots,\rho\}$ is a complete invariant under ***R***[s]-equivalence. The roots of $f_i(s)$ (including multiplicities) define finite zeros of **T**(s). The structure of these zeros (multiplicities and groupings) is defined by factorising the $f_i(s)$ into irreducible factors over the real, or complex numbers; for every zero z we define the set of z-elementary divisors by grouping all factors with root at z. The set of elementary divisors (for all zeros) is also a complete invariant under ***R***[s]-equivalence. Note that although the Smith form under ***R***[s]-equivalence defines the finite zero structure (finite frequencies), it does not convey any information on the structure at infinity; an alternative form is required and will be considered in a later section.

With a polynomial, or rational matrix **T**(s) we may associate two important rational vector spaces (vector spaces of rational vectors and with scalars the rational functions ***R***(s)). These are:

$$\mathcal{N}_r(\mathbf{T}) = \{\mathbf{x}(s): \mathbf{T}(s)\,\mathbf{x}(s) = \mathbf{0},\ \mathbf{x}(s)\ \ r \times 1 \text{ vectors}\} \tag{13}$$
$$\mathcal{N}_\ell(\mathbf{T}) = \{\mathbf{y}^t(s): \mathbf{y}^t(s)\,\mathbf{T}(s) = \mathbf{0},\ \mathbf{y}(s)^t\ 1 \times q \text{ vectors}\}$$

$\mathcal{N}_r(\mathbf{T})$, $\mathcal{N}_\ell(\mathbf{T})$ are called respectively right-, left-rational vector spaces, $\dim\mathcal{N}_r(\mathbf{T})=r-\rho$ $\dim\mathcal{N}_\ell(\mathbf{T})=q-\rho$ and with such spaces we can always define polynomial bases. If **X**(s) is an $r \times (r-\rho)$ polynomial basis for $\mathcal{N}_r(\mathbf{T})$, or of any rational vector space $\mathcal{X}$ with dim $\mathcal{X} = r - \rho$, then it is called least degree if it has no zeros. A polynomial basis $\mathbf{X}(s) = [\mathbf{x}_1(s),\ldots,\mathbf{x}_{r-\rho}(s)]$ with column degrees $\{d_1,\ldots,d_{r-\rho}\}$ is called least complexity, if $\Sigma\, d_i = \delta(\mathbf{X})$; note that $\delta(\mathbf{X})$ denotes the degree of **X**(s), defined as the maximal of the degrees of all maximal order minors of **X**(s). A least degree and least complexity polynomial basis of $\mathcal{N}_r(\mathbf{T})$ is called a minimal basis, the ordered set of its degrees $\{d_1,..,d_{r-\rho}\}$ are called right minimal indices and $\delta_r(\mathbf{T}) = \Sigma\, d_i$ is the right-order of **T**(s). The notion of left minimal indices and left order are defined similarly on $\mathcal{N}_\ell(\mathbf{T})$. Note that the sets of minimal indices are invariants of the corresponding rational vector spaces (but not complete).

A special case of a polynomial matrix is that of a matrix pencil s**F**-**G**, where **F**,**G** are qxr real (or complex) matrices and s is an independent complex variable taking values on the compactified complex plane (that includes the point at infinity). For such matrices we define the notion of strict equivalence in the following way: Two pencils s**F**-**G**, $sF' - G'$ of dimension qxr are strict equivalent, if there exist real matrices **Q**, **R** of dimension qxq, rxr respectively such that

$$sF' - G' = \mathbf{Q}\,(s\mathbf{F}-\mathbf{G})\,\mathbf{R}, \quad |\mathbf{Q}|, |\mathbf{R}|, \neq 0 \tag{14}$$

The above introduces the notion of strict equivalence of matrix pencils and the equivalence classes are characterized by a set of invariants that will be defined subsequently. Pencils may be represented in a homogenous form as $sF' - \hat{s}G'$ where s, $\hat{s}$ are independent complex variables. Frequencies on the compactified complex plane are represented as ordered pairs (α,β), where at least one of the α, β is $\neq 0$. Pairs (α,β): $\beta \neq 0$ correspond to finite frequencies. With the homogeneous pencil $s\mathbf{F} - \hat{s}\mathbf{G}$ we associate the single variable pencils $s\mathbf{F} - \mathbf{G}$, $s\mathbf{F} - \hat{s}\mathbf{G}$. The sets of invariants that may be defined are:

Strict Equivalence Invariants of Matrix Pencils.

Elementary Divisors: The Smith form of the homogeneous pencil $s\mathbf{F} - \hat{s}\mathbf{G}$ defines a set of elementary divisors of the following type: s^p, $(s - \alpha\hat{s})^\tau$, $\hat{s}^q$. The set of elementary divisors s^p, $(\hat{s} - \alpha)^\tau$, are called finite elementary divisors (fed) of $s\mathbf{F} - \mathbf{G}$, whereas those of the $\hat{s}^q$ type are called infinite elementary divisors (ied) of $s\mathbf{F} - \mathbf{G}$.

Minimal Indices: A matrix pencil $s\mathbf{F}-\mathbf{G}$, where at least one of $\mathcal{N}_r\{s\mathbf{F}-\mathbf{G}\}$ or $\mathcal{N}_\ell\{s\mathbf{F}-\mathbf{G}\}$ are non trivial $(\neq 0)$ are called singular, otherwise they are regular. If $\mathcal{N}_r\{s\mathbf{F}-\mathbf{G}\}$ $\{\neq 0\}$, then the minimal indices of this space are denoted $\boldsymbol{I_c}$ **(F, G) =** $\{\varepsilon_i, i=1,\dots,\mu\}$ and are referred to as column minimal indices (*cmi*) of the pencil. Similarly, if $\mathcal{N}_\ell\{s\mathbf{F}-\mathbf{G}\}$ $\neq \{0\}$ then $\boldsymbol{I_r}(\mathbf{F}, \mathbf{G}) = \{\eta_j, j=1,\dots,\nu\}$ and are referred to as row minimal indices *(rmi).*

This set of invariants is complete for the strict equivalence of matrix pencils, that is they uniquely characterize the strict equivalence class of a matrix pencil. There is a uniquely defined element by the invariants referred to as Kronecker canonical form.

Kronecker Canonical Form of a matrix Pencil: Consider a matrix pencil $s\mathbf{F} - \mathbf{G}$ and assume that its Kronecker invariants are: elementary divisors of the type: $\{(s-\alpha)^\tau, \dots; \hat{s}^q\}$ column minimal indices: $\{\varepsilon_1 = \dots = \varepsilon_t = 0,\ \varepsilon_j > 0, j = t+1, \dots\mu\}$, row minimal indices: $\{\eta_1 = \dots = \eta_\sigma = 0\ \ \eta_i > 0,\ i = \sigma + 1,\dots,\nu\}$. There always exists a pair of strict equivalence transformations $\mathbf{Q}$, $\mathbf{R}$ such that

$$\mathbf{Q}\,(s\mathbf{F} - \mathbf{G})\,\mathbf{R} = \text{block diag}\,\{\,\mathbf{O}_{\sigma,\,t};\dots\mathbf{L}_\varepsilon(s)\,,..,\mathbf{L}_\eta(s)_{,\dots;}\ s\,\mathbf{F}_w - \mathbf{G}_w\}$$

$$\mathbf{L}_\varepsilon(s) = s\,[\mathbf{I}_\varepsilon\ \mathbf{0}] - [\mathbf{0}, \mathbf{I}_\varepsilon] :\ \varepsilon\text{x}\ (\varepsilon+1)\text{ block}$$

$$\mathbf{L}_\eta(s) = s\begin{bmatrix} I_\eta \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ I_\eta \end{bmatrix} : (\eta + 1)\text{ x }\eta\text{ block} \tag{15}$$

$$s\mathbf{F}_w - \mathbf{G}_w = \text{block . diag}\,\{s\mathbf{I}\text{-}\mathbf{J}(a);\ \dots\ s\mathbf{H}_q - \mathbf{I}_q\dots\}$$

where $\mathbf{J}(a)$ is the τx τ Jordan block associated with $(s-\alpha)^\tau$ and $\mathbf{H}_q$ is a qxq nilpotent block (1s on the first super diagonal and the rest zero).

4. Finite Poles and Zeros of State Space Models: Dynamics and their Geometry

4.1. Eigenvalues, Eigenvectors and Free Rectilinear Motions.

For a single input, single output (SISO) system represented by a rational transfer function g(s) where g(s) = n(s)/d(s) and n(s), d(s) are coprime polynomials with deg{n(s)}=r and deg {d(s)} = n we define as finite poles the roots of d(s) and as finite zeros the zeros of n(s). If r< n we say that g(s) has an infinite zero of order n – r, and if r > n then g(s) has a infinite pole with order n – r. Finite and infinite poles have the property that the gain of the transfer function becomes unbounded (tends to infinity) and finite and infinite zeros are those frequencies for which the gain vanishes. In this sense, the notion of poles and zeros are dual since the first characterizes resonance and the second anti-resonances. It is this basic property that motivates a number of the definitions and problems that relate to multivariable poles and zeros.

For a state space model the internal natural dynamics are defined by the zero input response ($\mathbf{u}(t) = \mathbf{0}$), which in turn is characterized by the eigenvalues and eigenvectors of the state matrix **A** (see *System Description in Time Domain*), which determine the solution space of the autonomous system. Using $S(\Phi,\Omega)$ description for $\mathbf{u}(s)\equiv 0$ we get the zero input differential description.

$$\mathbf{S}(\mathbf{H},\boldsymbol{\Theta}):\begin{bmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{\mathbf{y}}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{C} & -\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{bmatrix} \tag{16}$$

which of course is equivalent to

$$\dot{x}(t) = \mathbf{A}\,\mathbf{x}(t),\quad \mathbf{y} = \mathbf{C}\,\mathbf{x}(t) \tag{17}$$

Definition 1: We define as poles of the state space model the eigenvalues of **A** and as *pole directions* the corresponding eigenvectors.

If (λ, **u**) is a pair of an eigenvalue and eigenvector of **A**, then for every initial condition $\mathbf{x}(0) = c\mathbf{u}$ (c constant) the corresponding solution of $\mathbf{S}(\mathbf{H},\boldsymbol{\Theta})$ is

$$\mathbf{x}(t) = \exp(\lambda t)\, c\mathbf{u},\quad \mathbf{y}(t) = \exp(\lambda t)\ \mathbf{Cu} \tag{18}$$

which are λ exponential trajectories in the state and output spaces along the constant direction $\mathbf{x}_\lambda = \mathbf{u}$ and $\mathbf{y}_\lambda = \mathbf{Cu}$ Such straight line exponential motions are frequently called rectilinear motions and $\mathbf{x}_\lambda$, $\mathbf{y}_\lambda$ are called state pole and output pole directions correspondingly. The fundamental operator describing the pole structure is the pencil $\mathbf{S}(\mathbf{H},\boldsymbol{\Theta})$ defined from the description (1), or in a reduced form the pencil

$$\mathbf{W}(s) = s\mathbf{I} - \mathbf{A} \tag{19}$$

which is called the state pole pencil. The eigenstructure of **A** is defined by its Jordan form (see *Canonical Forms for State Space Descriptions*) and this may be defined algebraically by the elementary divisors of the Smith form of **W**(s). The presence of elementary divisors with degree higher than one implies the existence of Jordan blocks for the corresponding eigenvalues. The dynamic characterization of such multiple eigenvalues is in terms of generalized rectilinear motions (involving terms of the type $\exp(\lambda t)t^{i}, i = 1,2....$).

The pole-zero duality motivates the definition of the output zeroing problem, that is investigation the type of solutions of the system for which the output is identically zero (the output now is zero instead of the input for the case of pole). When **y**(t) ≡ **0** the **S(Φ, Ω)** description is reduced to the system

$$S(\Gamma,\Delta): \begin{bmatrix} I & O \\ O & O \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{20}$$

the solutions of which describe the zero dynamics of the original system. The fundamental pencil of **S**(Γ, Δ) is the Rosenbrock system matrix pencil and its structure describe the zero structure of the system. The dynamic characterization of the zero structure benefits by extending first the notion of free rectilinear motion to that of the forced rectilinear motion.

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