FREQUENCY DOMAIN REPRESENTATION AND SINGULAR VALUE DE-COMPOSITION

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Contents

Summary

This chapter reviews the external and the internal representations of linear-time-invariant systems. This is done both in the time and the frequency domains. The realization problem is then discussed. Given the importance of norms in control design and model reduction, the final part of this chapter is dedicated to the definition and computation of various norms. Again, the interplay between time and frequency norms is emphasized.

1. Introduction

One of the most powerful tools in the analysis and synthesis of linear time-invariant systems is the equivalence between the *time domain* and the *frequency domain*. Thus additional insight into problems in this area is obtained by viewing them both in *time* and in *frequency*. This dual nature accounts for the presence and great success of linear systems both in engineering theory and applications.

In this chapter we will provide an overview of certain results concerning the analysis of linear dynamical systems. Time and frequency domain frameworks are inextricably connected. Therefore together with frequency domain considerations in the sequel, unavoidably, a good deal of time domain considerations are included as well.

Our goals are as follows. First, basic system representations will be introduced, both in time and in frequency. Then the ensuing *realization problem* is formulated and solved. Roughly speaking the realization problem entails the construction of a state space model from frequency response data.

The second goal is to introduce various *norms* for linear systems. This is of great importance both in *control design* and in *system approximation/model reduction*.

First it is shown that besides the convolution operator we need to attach a second operator to every linear system, namely the Hankel operator. The main attribute of this operator is that it has a discrete set of singular values, known as the *Hankel singular values*. These singular values are main ingredients of numerous computations involving control design and model reduction of linear systems. Besides the Hankel norm, we discuss various p-norms, where $p = 1, 2, \infty$. It turns out that norms which are obtained for $p = 2$ have both a time domain and a frequency domain interpretation. The rest have an interpretation in the time domain only.

The chapter is organized as follows. The next section is dedicated to a collection of useful result on two topics: The Laplace and discrete-Laplace transforms on the one hand and norms and the SVD on the other. Tables 1 and 2, summarize the salient properties of these two transforms. Section 3 develops the external and internal representations of linear systems. This is done both in the time and frequency domains, with the results summarized in two further Tables 3,4. This discussion is followed by the formulation and solution of the realization problem. Finally Section 4 is dedicated to the exposition of various norms for linear systems. The basic features of these norms are summarized in Table 5. We conclude by outlining the use of the various norms in control system design and system approximation.

2. Preliminaries

2.1. The Laplace Transform and the $\mathscr{Z}$ -transform

The logarithm can be considered as an elementary transform. It assigns a real number to any positive real number. It was invented in the middle ages and its purpose was to convert the *multiplication* of multi-digit numbers to addition. In the case of linear, time-invariant systems the operation which one wishes to simplify is the *derivative with respect to time* in the continuous-time case or the *shift* in the discrete-time case. As a consequence, one also wishes to simplify the operation of *convolution*, both in discrete- and continuous-time.

Thus an operation is sought which will transform derivative into simple multiplication in the transform domain. In order to achieve this however, the transform needs to operate on functions of time. The resulting function will be one of complex frequency. This establishes two equivalent ways of dealing with linear, time-invariant systems, namely in the *time domain* and in the *frequency domain*. In the next two sections we will briefly review some basic properties of this transform, which is called Laplace transform in continuous-time and discrete-Laplace of $\mathscr{Z}$ -transform in discrete-time.

2.1.1. Some Properties of the Laplace Transform

Consider a function of time $f(t)$. The *unilateral Laplace transform* of f is a function denoted by $F(s)$ of the complex variables $s = \sigma + jw$. The definition of F is as follows:

$$f(t) \overset{\mathcal{L}}{\rightarrow} F(s) := \int_{0-}^{\infty} f(t) e^{-st} dt \tag{2.1}$$

Therefore the values of f for negative time are ignored by this transform. Instead, in order to capture the influence of the past, initial conditions at time zero are required (see *Differentiation in time*, see Table 1). The salient properties of the Laplace transform are summarized in Table 1.

2.1.2. Some Properties of the $\mathscr{Z}$ -transform

Consider a function of time $f(t)$, where time is discrete $t \in \mathbb{Z}$. The *unilateral* $\mathscr{Z}$ *-transform* of f is a function denoted by $F(z)$ of the complex variable $z = re^{j\theta}$. The definition of F is as follows:

$$f(t) \xrightarrow{\mathcal{Z}} F(z) := \sum_{t=0}^{\infty} z^{-t} f(t) \tag{2.2}$$

The main features of this transform are summarized in Table 2.

Basic Laplace transform properties		
Property	Time signal	$\mathcal{L}$*-transform*
Linearity	$af_1(t) + bf_2(t)$	$aF_1(s) + bF_2(s)$
Shifting in the s-domain	$e^{s_0 t} f(t)$	$F(s - s_0)$
Time scaling	$f(at), a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Convolution	$f_1(t) * f_2(t)$ $f_1(t) = f_2(t) = 0, t < 0$	$F_1(s)F_2(s)$
Differentiation in time	$\frac{d}{dt} f(t)$	$sF(s) - f(0^-)$
Differentiation if freq.	$-tf(t)$	$\frac{d}{ds} F(s)$
Integration in time	$\int_{0^-}^{t} f(\tau) d\tau$	$\frac{1}{s} F(s)$
Impulse	$\delta(t)$	*1*
Exponential	$e^{at}\mathbb{I}(t)$	$\frac{1}{s-a}$
Initial value Theorem : $f(0^+) = \lim_{s\to\infty} sF(s)$		
Final value theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$		

Table 1: Basic Laplace transform properties (The last 2 properties hold provided that $f(t)$ contains no impulses or higher-order singularities at $t = 0$.)

Basic $\mathcal{Z}$ -transform properties $(t \in \mathbb{Z})$

Property	Time signal	$\mathscr{Z}$ -transform
Linearity	$af_1(t)+bf_2(t)$	$aF_1(z)+bF_2(z)$
Forward shift	$f(t-1)$	$z^{-1}F(z)+f(-1)$
Backward shift	$f(t+1)$	$z^{-1}F(z)+f(-1)$
Scaling in freq.	$a^t f(t)$	$F\left(\frac{z}{a}\right)$
Conjugation	$f^*(t)$	$F^*\left(z^*\right)$
Convolution	$f_1(t)* f_2(t)$ $f_1(t)=f_2(t)=0, t<0$	$F_1(z)F_2(z)$
Differentiation if freq.	$tf(t)$	$-z\frac{d}{dz}F(z)$
Impulse	$\delta(t)$	1
Exponential	$a^n\mathbb{I}(t)$	$\frac{z}{z-a}$
First difference	$f(t)-f(t-1)$	$(1-z^{-1})F(z)-f(-1)$
Accumulation	$\sum_{k=0}^{n} f(t)$	$\frac{1}{1-z^{-1}}F(z)$
Initial value Theorem : $f[0]=\lim_{z\to\infty} F(z)$		
Final value theorem: $\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$		

Table 2: Basic $\mathscr{Z}$ -transform properties

2.2. Norms of Vectors, Matrices and the SVD

In this section we will first review some material from liner algebra which pertains to norms of vectors, norms of operators (matrices), both in finite and infinite dimensions. The latter are of importance because a linear system can be viewed as a map between infinite dimensional spaces. The Singular Value Decomposition (SVD) will also be introduced and its properties briefly discussed.

2.2.1. Norms of Finite-dimensional Vectors and Matrices

Let X be a linear space over the field $\mathbb{K}$ which is either the field of reals $\mathbb{R}$ or that of complex numbers $\mathbb{C}$. A norm on X is a function $v : X \to \mathbb{R}$, such that the following three properties are satisfies. Non-strict positivity: $v(x) \geq 0, \ \forall x \in X$, with equality if $x = 0$; triangle inequality: $v(x+y) \leq v(x) + v(y), \ \forall x, y \in X$; positive homogeneity: $v(\alpha x) = |\alpha| v(x), \ \forall \alpha \in \mathbb{K}, \ \forall x \in X$. For vectors $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$ the **Hölder** or p **-norms** are defined as follows:

$$\| x \|_p \quad := \begin{cases} (\Sigma_{i \in \underline{n}} \, | x_i |^p)^{\frac{1}{p}}, \ 1 \leq p < \infty \\ \quad \max_{i \in \underline{n}} | x_i |, \ p = \infty \end{cases} , \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \tag{2.3}$$

where $\underline{n} := \{1, 2, \ldots, n\}, n \in \mathbb{N}$. The 2-norm satisfies the *Cauchy-Schwartz inequality*:

$$| x^* y | \leq \| x \|_2 \| y \|_2$$

with equality holding if and only if $y = cx, \ c \in \mathbb{K},$ or $y = 0$. An important property of the 2-norm is that it is invariant under *unitary* (orthogonal) transformations; let U be such a transformation, that is, $UU^* = U^* U = I_n$. It follows that $\| Ux \|_2^2 = x^* U^* U x = x^* x = \| x \|_2^2$. The following relationship between the Hölder norms for $p = 1, 2, \infty$ holds:

$$\| x \|_\infty \ \leq \ \| x \|_2 \ \leq \ \| x \|_1$$

One type of matrix norms are those which are *induced* by the vector p -norms defined above.

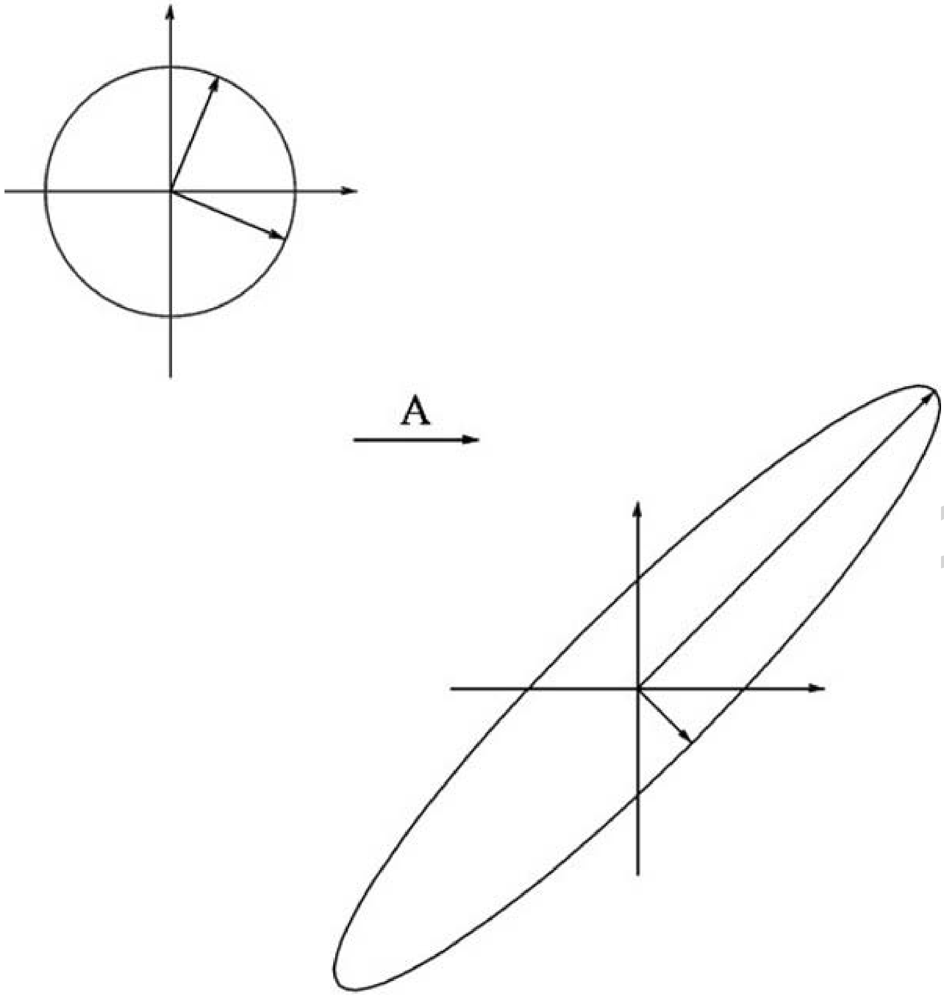

Figure 1: The linear transformation A maps the unit sphere into an ellipsoid. The singular values are the lengths of the semi-axes of the ellipsoid.

More precisely for $A \in \mathbb{C}^{m\times n}$

$$\| A\|_{p,q-\mathrm{ind}} := \sup_{x\neq 0} \frac{\| Ax\|_q}{\| x\|_p} \tag{2.4}$$

is the *induced* p,q*-norm* of A. In particular, for $p = q = 1, 2, \infty$ the following expressions hold

$$\| A\|_1 = \max_{i\in \underline{n}} \sum_{j\in \underline{m}} | A_{ij} |,\ \| A\|_\infty = \max_{j\in \underline{m}} \sum_{i\in \underline{n}} | A_{ij} |,\ \| A\|_2 = [\lambda_{max}(AA^*)]^{\frac{1}{2}}$$

Besides the induced matrix norms, there exists other norms. One such class is the *Schatten* p*-norms* of matrices. These non-induced norms are unitarily invariant. Let $\sigma_i(A),\ 1 \le i \le \min(m,n)$, be the singular values of A, i.e. the square roots of the eigenvalues of AA^* (see also Section 2.2.2). Then

$$\|A\|_p := \left(\sum_{i\in \underline{m}} \sigma_i^p(A)\right)^{\frac{1}{p}}, \quad 1\le p<\infty \tag{2.5}$$

It follows that the Schatten norm for $p=\infty$ is

$$\|A\|_\infty = \sigma_{max}(A)$$

which is the same as the 2-induced norm of A. For $p=1$ we obtain the *trace norm*

$$\|A\|_1 = \sum_{i\in \underline{m}} \sigma_i(A)$$

For $p=2$ the resulting norm is also known as the **Frobenius norm**, the **Schatten 2-norm**, or the **Hilbert-Schmidt norm** of A:

$$\|A\|_{\mathrm{F}} = \left(\sum_{i\in \underline{m}} \sigma_i^2(A)\right)^{\frac{1}{2}} = (\mathrm{trace}(A^*A))^{\frac{1}{2}} = (\mathrm{trace}(AA^*))^{\frac{1}{2}} \tag{2.6}$$

where trace $(\cdot)$ denotes the *trace* of a matrix

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Biographical Sketch

Thanos Antoulas was born in Athens, Greece. He studied at the ETH Zürich, where he obtained the Diploma of Electrical Engineering in 1975, the Diploma of Mathematics in 1975, and the Ph.D. Degree in Mathematics in 1980. He was Professor R.E. Kalman's only Ph.D. student in Switzerland.
Since 1982 he has been with the Department of Electrical and Computer Engineering, Rice University, where he is currently a Professor.

Professor Antoulas was elected Fellow of the IEEE (Institute of Electrical and Electronics Engineers) in 1991 and was awarded the Best Paper Prize of the AIAA (American Institute of Aeronautics and Astronautics) Guidance, Navigation and Control Section, in 1992. In 1995 he was awarded a JSPS (Japan Society for the Promotion of Science) Fellowship. He has served on the Editorial Board of many journals including the IEEE Transactions on Automatic Control, SIAM Journal on Control and Optimization, and Linear Algebra and its Applications. For the past 8 years he has been serving as Editor-in-Chief of Systems and Control Letters.

He has held many visiting appointments, including those at the Australian National University, the University of Groningen, the Catholic University of Leuven and Louvain-la-Neuve, the Tokyo Institute of

Technology, Kyoto University, etc. Furthermore he has given invited presentations at numerous conferences and workshops. His research interests are in the broad area of dynamical systems and computation including model reduction of large-scale systems. He is the author of a book on the Approximation of large-scale systems, published by SIAM.