POLYNOMIAL AND MATRIX FRACTION DESCRIPTION

Didier Henrion
Laboratoire d'Analyse et d'Architecture des Systèmes, Centre National de la Recherche Scientifique, Toulouse, France.

Michael Šebek
Center for Applied Cybernetics, Faculty of Electrical Engineering, Czech Technical University in Prague, Czech Republic.

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Contents

Summary

This article illustrates how polynomials and polynomial matrices can be used to describe linear systems. The focus is put on the transformation to and from the state-space equations, because it is a convenient way to introduce gradually the most important properties of polynomials and polynomial matrices, such as: coprimeness, greatest

common divisors, unimodularity, column-and row-reducedness, canonical Hermite or Popov forms.

1. Introduction

The first step when studying and designing a control strategy for a physical system is the development of mathematical equations that describe the system. These equations are obtained by applying various physical laws such as Kirchoff's voltage and current laws (electrical systems) or Newton's law (mechanical systems).

The equations that describe the physical system may have different forms. They may be linear equations, nonlinear equations, integral equations, difference equations, differential equations and so on. Depending on the problem being treated, one type of equation may prove more suitable than others. The linear equations used to describe linear systems are generally limited either to:

- the input-output description, or external description in the frequency domain, where the equations describe the relationship between the system input and system output in the Laplace transform domain (continuous-time systems) or in the z-transform domain (discrete-time systems), or

- the state-variable equation description, or internal description, a set of first-order linear differential equations (continuous-time systems) or difference equations (discrete-time systems).

Prior to 1960, the design of control systems had been mostly carried out by using transfer functions. However, the design had been limited to the single variable, or single-input-single-output (SISO) case. Its extension to the multivariable, or multi-input-multi-output (MIMO) case had not been successful.

The state-variable approach was developed in the sixties, and a number of new results were established in the SISO and MIMO cases. At that time, these results were not available in the transfer-function, or polynomial approach, so the interest in this approach was renewed in the seventies. Now most of the results are available in both the state-space and polynomial settings.

The essential difference between the state-space approach and the polynomial approach resides in the practical way control problems are solved. Roughly speaking, the state-space approach heavily relies on the theory of real and complex matrices, whereas the polynomial approach is based on the theory of polynomials and polynomial matrices.

For historical reasons, the computer-aided, control-system, design packages have been mostly developed in the late eighties and nineties for solving control problems formulated in the state-space approach.

Polynomial techniques, generally simpler in concepts, were most notably favored by lecturers teaching the basics of control systems, and the numerical aspects have been left aside. Recent results tend however to counterbalance the trend and several reliable and efficient numerical tools are now available to solve problems involving polynomials and polynomial matrices. In particular, the Polynomial Toolbox for Matlab is recommended for numerical computations with polynomials and polynomial matrices. Polynomial matrices, i. e. matrices with polynomial entries, can be found in a variety of applications in science and engineering.

Second degree polynomial matrices arise in the control of large flexible space structures, earthquake engineering, the control of mechanical multi-body systems, stabilization of damped gyroscopic systems, robotics, and vibration control in structural dynamics. For illustration, natural modes and frequencies of a vibrating structure such as the Millennium footbridge over the river Thames in London are captured by the zeros of a quadratic polynomial matrix.

Third degree polynomial matrices are sometimes used in aero-acoustics. In fluid mechanics, the study of the spatial stability of the Orr-Sommerfeld equation yields a quartic matrix polynomial. In this article, we describe a series of concepts related to polynomial matrices. We introduce them gradually, as they naturally arise, when studying standard transformations to and from the state-space domain.

2. Scalar Systems

2.1. Rational Transfer Function

The transfer function description of a system gives a mathematical relation between the input and output signals of the system. Assuming zero initial conditions, the relationship between the input u and the output y of a system can be written as $[y(s) = G(s)u(s)]$ where s is the Laplace transform in continuous-time (for discrete-time systems, we use the z-transform and the variable z), and $G(s)$ is the scalar *transfer function* of the system. $G(s)$ is a rational function of the indeterminate s that can be written as a ratio of two polynomials $[G(s) = \frac{n(s)}{d(s)}]$ where $n(s)$ is a *numerator polynomial* and $d(s)$ is a *denominator polynomial* in the indeterminate s.

In the above description of a transfer function, it is assumed that polynomials $n(s)$ and $d(s)$ are *relatively prime*, or *coprime* polynomials, i. e. they have no common factor, except possibly constants.

The degree of denominator polynomial $d(s)$ is the order of the linear system. When the denominator polynomial is *monic*, i.e. with leading coefficient equal to one, the transfer function is *normalized* or *nominal*.

It is always possible to normalize a transfer function by dividing both numerator and denominator polynomials by the leading coefficient of the denominator polynomial.

As an example, consider the mechanical system shown in Figure 1.

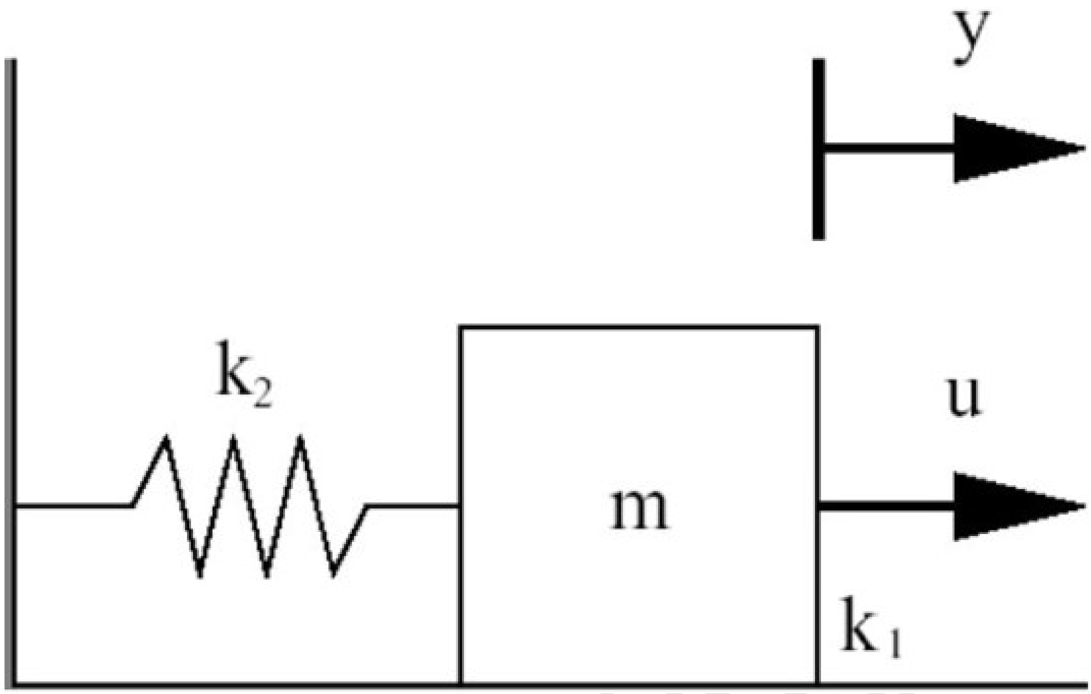

Figure 1: Mechanical system.

For simplicity, we consider that the friction force between the floor and the mass consists of viscous friction only (we neglect the static friction and Coulomb friction). It is given by $f = k_1 dy/dt$, where k_1 is the viscous friction coefficient. We also assume that the displacement of the spring is small, so that the spring force is equal to k_2y, where k_2 is the spring constant.

Applying Newton's law, the input-output description of the system from the external force u (input) to the displacement y (output) is given by $[m\frac{d^2y}{dt^2} = u - k_1\frac{dy}{dt} - k_2 y]$ Taking the Laplace transform and assuming zero initial conditions, we obtain $[ms^2y(s) = u(s) - k_1sy(s) - k_2y(s)]$ so that $[y(s) = \frac{1}{ms^2+k_1s+k_2}u(s) = G(s)u(s)]$.

Transfer function $G(s)$ has numerator polynomial $n(s) = 1$ of degree zero and denominator polynomial $d(s) = ms^2 + k_1s + k_2$ of degree two. The corresponding linear system has therefore order two. Dividing both $n(s)$ and $d(s)$ by the leading coefficient of $d(s)$ we obtain the normalized transfer function $[G(s) = \frac{\frac{1}{m}}{s^2+\frac{k_1}{m}s+\frac{k_2}{m}}]$.

2.2. From Transfer Function To State-Space

Similarly to network synthesis where the objective is to build a network that has a prescribed impedance or transfer function, it is very useful in a control system design to determine a dynamical equation that has a prescribed rational transfer matrix $G(s)$. Such an equation is called a *realization* of $G(s)$. The most common ones for linear systems are state-space realizations of the form

$$\begin{bmatrix} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{bmatrix} \tag{1}$$

where $x(t)$ is the state vector, $u(t)$ is the input, $y(t)$ is the output and A, B, C are matrices of appropriate dimensions. Such realizations correspond to strictly proper transfer functions. In the case of proper transfer function, one must add a direct transmission term $Du(t)$ to the output variable $y(t)$.

For simplicity, we shall assume that $D = 0$ in the sequel. For every transfer function $G(s)$, there is an unlimited number of state-space realizations. Therefore, it is relevant to introduce some commonly used, or *canonical* realizations.

We shall present two of them in the sequel: the controllable form and the observable form. However, note there are other canonical forms such as the controllability, observability, parallel, cascade or Jordan form, that we do not describe here for conciseness.

2.2.1. Controllable Canonical Form

For notational simplicity, we consider a system of third order, with normalized strictly proper transfer function $[G(s) = \frac{n(s)}{d(s)} = \frac{n_0 + n_1 s + n_2 s^2}{d_0 + d_1 s + d_2 s^2 + s^3}]$. One can then easily extend the results to systems of arbitrary order. The controllable canonical realization corresponding to $G(s)$ has state-space matrices:

$$\begin{bmatrix} A = \begin{bmatrix} -d_2 & -d_1 & -d_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & C = \begin{bmatrix} n_2 & n_1 & n_0 \end{bmatrix} \end{bmatrix}. \tag{2}$$

As its name suggests, this realization is always controllable no matter whether $n(s)$ and $d(s)$ are coprime or not. If $n(s)$ and $d(s)$ are coprime, then the realization is observable as well.

2.2.2. Observable Canonical Form

The observable canonical realization corresponding to $G(s)$ has state-space matrices

$$\begin{bmatrix} A = \begin{bmatrix} -d_2 & 1 & 0 \\ -d_1 & 0 & 1 \\ -d_0 & 0 & 0 \end{bmatrix} & B = \begin{bmatrix} n_2 \\ n_1 \\ n_0 \end{bmatrix} & C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{bmatrix}. \tag{3}$$

Note that this realization is dual to the controllable canonical realization in the sense that matrix A is transposed, and vectors B and C are interchanged and transposed. Obviously, this form is always observable. If $n(s)$ and $d(s)$ are coprime, it is also controllable.

2.3. From State-Space To Transfer Function

Assuming zero initial conditions and taking the Laplace transform of the state-space equations we obtain that $[G(s) = C(sI - A)^{-1}B]$ where I denotes the identity matrix of the same dimension as matrix A.

Recalling the formula of the inverse of a matrix, the above equation can be written as

$[C(sI - A)^{-1}B = \frac{C\mathrm{adj}(sI-A)B}{\det(sI-A)} = \frac{\bar{n}(s)}{\bar{d}(s)}]$.

Polynomial $\bar{d}(s)$ is generally referred to as the *characteristic polynomial* of matrix A. It may happen that polynomials $\bar{n}(s)$ and $\bar{d}(s)$ have some common factors captured by a common polynomial term $f(s)$, so that we can write $[\frac{\bar{n}(s)}{\bar{d}(s)} = \frac{n(s)f(s)}{d(s)f(s)} = \frac{n(s)}{d(s)}]$ where $n(s)$ and $d(s)$ are coprime.

The ratio of $n(s)$ over $d(s)$ as defined above is a representation of the transfer function $G(s)$. When $n(s)$ and $d(s)$ are coprime the representation is called *irreducible*. It turns out that $G(s)$ is irreducible if and only if pair (A, B) is controllable and pair (C, A) is observable (see *System Characteristics: Stability, Controllability, Observability*).

Checking the relative primeness of two polynomials $n(s)$ and $d(s)$ can be viewed as a special case of finding the *greatest common divisor* (GCD) of two polynomials. This can be done either with the Euclidean division algorithm, or with the help of Sylvester matrices.

2.4. Minimality

A state-space realization (A, B, C) of a transfer function $G(s)$ is *minimal* if it has the smallest number of state variables, i. e. matrix A has the smallest dimension. It can be proven that (A, B, C) is minimal, if and only if the two polynomials defined above $\bar{n}(s) = C\mathrm{Adj}\,(sI - A)B$ and $\bar{d}(s) = \det\,(sI - A)$ are coprime, or equivalently, if and only if (A, B) is controllable and (C, A) is observable (see *System Characteristics: Stability, Controllability, Observability*).

3. Multivariable Systems

When trying to extend for multivariable systems the results on scalar systems presented in the previous section, several difficulties must be overcome. Multivariable systems are more involved because, unlike the scalar case, there does not seem to be a single unique canonical choice of realizations. Moreover, the connection with irreducible transfer functions is not obvious.

The closest analogy with the scalar results can be achieved by using the so-called *matrix fraction descriptions* (MFDs) of rational matrices as the ratio of two relatively-prime polynomial matrices. To handle these objects, several properties of polynomial matrices must be introduced.

3.1. Matrix Fraction Description

By analogy with the scalar case, a given rational matrix $G(s)$ can be written as a fraction of two polynomial matrices, i. e. matrices with polynomial entries.

As the product of matrices is not commutative, there exist two different ways to proceed. We can define a *right matrix fraction description*, or right MFD for short,

$$G(s) = N_R(s)D_R^{-1}(s) \tag{4}$$

where non-singular polynomial matrix $D_R(s)$ enters $G(s)$ from the right. Here non-singularity of a polynomial matrix means that its determinant is not identically zero, or equivalently that the matrix is non-singular for almost all values of the indeterminate. For example, the matrix

$$\begin{bmatrix} 1 & s \\ s+1 & s^2+1 \end{bmatrix} \tag{5}$$

is non-singular, whereas the matrix

$$\begin{bmatrix} 1 & s \\ s+1 & s^2+s \end{bmatrix} \tag{6}$$

is singular. Alternatively, we can also define a *left MFD*

$$G(s) = D_L^{-1}(s)N_L(s) \tag{7}$$

where now the denominator polynomial matrix enters $G(s)$ from the left. As an example of a left

MFD, we consider the RCL network depicted on Figure 2, where the system outputs are the voltage and current through the inductor, and the input is the voltage.

Applying Kirchoff's laws, the Laplace transform and assuming zero initial conditions, we obtain the relation

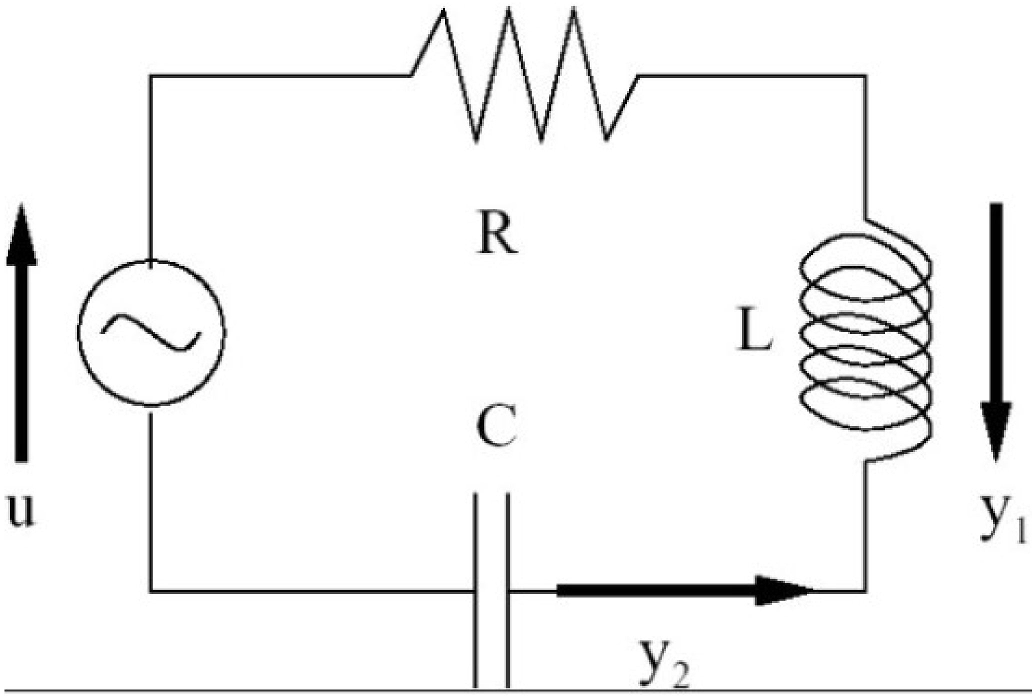

Figure 2: RCL network.

$$\begin{bmatrix} 1 & -Ls \\ Cs & 1+RCs \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ Cs \end{bmatrix} u \tag{8}$$

which defines the system transfer function matrix as a left MFD

$$G(s) = D_L^{-1}(s)N_L(s) = \begin{bmatrix} 1 & -Ls \\ Cs & 1+RCs \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ Cs \end{bmatrix}. \tag{9}$$

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Biographical Sketches

Michael Šebek was born in Prague, Czechoslovakia, in 1954. He obtained his Ing. degree in electrical engineering with distinction from the Czech University of Technology, Prague in 1978, his CSc. degree in control engineering and his DrSc. degree in control theory from the Czech Academy of Sciences in 1981 and 1995, respectively. He held visiting positions at the Strathclyde University, Glasgow, UK; Universiteit Twente, NL; and at ETH Zurich, CH.

Michael Šebek is Head of Control Engineering Department and Manager of the Center of Applied Cybernetics at the Czech Technical University in Prague. He is also Head of Control Theory Department in the Institute of Information Theory and Automation of the Czech Academy of Sciences and Chief Executive Officer of PolyX, Ltd. Prague.

His research interests include linear systems and signals, robust control, algorithms and software for control and filter design. He published well over 160 research papers and contributed to 10 books. Science Citation Index lists 170 citations of his works.

Michael Šebek has led numerous international and national research projects. In particular, he has coordinated EUROPOLY, the European Network of Excellence for Industrial Applications of Polynomial Methods.

He co-founded PolyX Ltd., Prague (www.polyx.com), the company producing software for systems, signals and control based on polynomial methods. He is currently the CEO of PolyX and leads the development of the company main product: the Polynomial Toolbox for Matlab.

Michael Šebek served as Associate Editor of the European Journal of Control. Within IFAC, he is vice-chair of the Policy Committee and member of the Technical Committee on Control Design. He is a Senior Member of the IEEE serving in the Czechoslovakia Section Executive Committee. He was the founding chairman of the Czech CSS Chapter. He is also a member of AMS, NYAS and SIAM.

Michael Šebek holds the Czech Government Prize for his achievements in algebraic control theory. He is listed in Marquis “Who's Who in the World and in Who's Who in Science and Engineering”.

For more info, visit www.michaelŠebek.cz .

Didier Henrion was born in Creutzwald, Northeastern France, in 1971. He received the "Diplôme d'Ingénieur" (Engineer's Degree) and the "Diplôme d'Etudes Approfondies" (Masters' Degree) with specialization in control from the Institut National des Sciences Appliquées (INSA, National Institute for Applied Sciences), Toulouse, Southwestern France, in September 1994. From October 1994 to December 1995, he was a research assistant at the Centro de Automatización Industrial of the Universidad Simón Bolívar, Caracas, Venezuela. From February 1996 to December 1998 he was a Ph.D. student at the Ustav Teorie Informace a Automatizace (UTIA, Institute of Information Theory and Automation), Prague, Czech

Republic.

He received his "C.Sc." (Ph.D.) degree from the Akademie Věd České Republiky (Academy of Sciences of the Czech Republic) in December 1998.

From October 1996 to August 1999 he was a Ph.D. student at the Laboratoire d'Analyse et d'Architecture des Systèmes (LAAS, Laboratory of Analysis and Architecture of Systems) of the Centre National de la Recherche Scientifique (CNRS, National Center for Scientific Research) in Toulouse, France. He received his "Diplôme de Doctorat" (Ph.D. degree) from INSA in October 1999. Since October 2000, he has been a "Chargé de Recherche au CNRS" (CNRS Researcher) at LAAS under the scientific responsibility of Jacques Bernussou. He is also holding a secondary appointment as a researcher at UTIA under the scientific responsibility of Michael Šebek. In October 2001 he was awarded a Young Researcher Grant by CNRS. Didier Henrion serves as an Associate Editor of the IFAC Automatica journal as well as an Associate Editor at the Conference Editorial Board of the IEEE Control Systems Society. He is vice-chair of the IFAC Technical Committee on Control Design. He is a regular member of the panel of reviewers of the American Mathematical Society and Zentralblatt Math.