OPTIMAL LINEAR QUADRATIC CONTROL

João Miranda Lemos
INESC-ID/IST, R. Alves Redol 9. 1000-029 Lisboa, Portugal.

Keywords: Optimal control, LQ, Quadratic performance index, Dynamic programming application, Riccati difference equation, Riccati differential equation, Algebraic Riccati equation, Chang-Letov design method, Root-square Locus, Polynomial control, Optimal State Feedback, Hamiltonian System, Pontryagin's Maximum Principle application, Kleinman's iterations, Cheap control.

Contents

Summary

The problem in which a linear plant is to be controlled such as to minimize a quadratic cost is addressed. Although this problem finds applications *per se*, *e. g.* in technology or Economic Planning, its main interest stems from being a basic block for the solution of other problems of major importance. A landmark example is provided by the LQG problem. The basic theory of the LQ regulator, in both continuous and discrete time, when the state of the plant to be controlled is available for direct measurement, is considered. The approach followed relies on the application of either Pontryagin's Maximum Principle or Bellman's Dynamic Programming. In continuous time the plant to control is assumed to be described by a linear controllable state space model. The resulting controller for the regulation problem consists of a state feedback whose gain K depends on the solution P of a non-linear matrix differential equation, called Riccati equation. For finite optimization intervals, P, and therefore the vector of feedback gains K, are time dependent. When the optimization horizon grows unbound, however,

the solution of the Riccati differential equation tends to a constant. The solution to the steady state LQ problem is thus given by a constant gain feedback control law, whose vector of gains depends on plant parameters, and a constant matrix P. This constant matrix satisfies the Riccati algebraic equation which results from equating to zero the time derivatives of P in the Riccati differential equations. Thus, the steady-state LQ controller is a specific type of state space. Under mathematical mild conditions (*e. g.* controllability of the state space realization considered), which amount to the problem being well posed in engineering terms (meaning that the right actuators and sensors are being used), the closed loop poles are such that the closed loop system is stable and possesses good stability margins. For discrete time plants the results parallel these. The structure of this article is as follows: After the introduction (section 1), which provides motivation and an overall view, the LQ regulator problem in continuous time is considered in section 2 for a finite optimization interval. Section 3 presents the resulting regulator in steady state, yielded by the limit when the optimization horizon is made larger and larger, the corresponding properties being considered in section 4. Section 5 parallels the theory presented in sections 2 to 4 for discrete time problems. Section 6 briefly considers numerical methods for the LQ problem. Finally, section 7 draws conclusions.

1. Introduction

LQ control refers to a problem in which a linear plant is to be controlled such as to minimize a quadratic cost. The following two examples help in elucidating about the problem to consider as well as the type of applications it might help solving.

Example 1.1

Choosing among alternative policies so as to best control the economy is among the up-most challenging and controversial problems facing humankind. Typical policy problems range from the choice of tax level to the way money supply should be adjusted, whether government expenditures should be increased or decreased or how Central Banks should adjust credit conditions. In an Automatic Control framework, these variables are seen as manipulated variables, affecting in a dynamic way other variables such as the rate of inflation, interest rates, the unemployment rate or the gross national product (GNP). The overall objective of the economic policy consists in stabilizing and regulating this system. This includes goals (selected according to Society common Will) such as the minimization of unemployment, the control of inflation, the desired rate of economic growth, the maintenance of a high level of investment or the redistribution of income through taxes and transfers.

The design of economic policies is considerably complicated due to aspects such as the dynamic structure of the economic system, its non-linear character and time variant features. Combined with the constraints imposed on both the state and manipulated variables, this results in a formidable optimization task.

If, however, short term optimization is considered, a linear model can be used yielding a tractable problem. For that sake, the control problem is structured so that the aim of the optimal policy plan is to make the system state vector $x(k)$ track as closely as possible a nominal state vector $x^n(k)$, subject to $u(k)$, the manipulated variables vector, track a nominal control vector $u^n(k)$. To say it in plain words, one would like variables such as the GNP, investment or unemployment to follow as closely as possible "ideal" time paths throughout the planning period.

In a mathematical setting this can be phrased as minimizing with respect to the sequence $\{u(k),\, 0 \le k \le N\}$ the functional

$$J = \frac{1}{2}\sum_{k=0}^{N}\left\{\left(x(k) - x^n(k)\right)' Q\left(x(k) - x^n(k)\right) + \left(u(k) - u^n(k)\right)' R\left(u(k) - u^n(k)\right)\right\}$$

where Q is an $n \times n$ positive semi-definite matrix, $n = \dim x$ and R is a positive definite matrix.

The functional J has two main parcels: The first one has to do with the state x and the other has to do with the manipulated variable u. Start by considering the first parcel. In the ideal case in which the state $x(k)$ exactly matches the nominal value $x^n(k)$ for all discrete times k, this parcel will be zero. In general this will not be possible and, at most, some positive (since the cost is quadratic) minimum value will be attained. Consider now the second parcel, which involves the manipulated variable, u. Again this puts a penalty on the deviations of $u(k)$ from its nominal value $u^n(k)$. In general, a kind of "water bed effect" takes place: Good tracking of $x^n(k)$ by $x(k)$ yields deviations of $u(k)$ with respect to $u^n(k)$ and vice-versa. The outcome of the optimization is a sequence of values of the manipulated variable u which "keeps the right balance" with respect to the trade-off of minimizing the sum composing J. Matrices Q and R allow to modify the relative importance given to one or the other parcel, as well as to the different entries of the vectors x and u, being thus important "knobs" for achieving solutions with the desired properties.

Example 1.2

Airplanes are provided of moving surfaces (such as ailerons, the rudder or rear wing deflectors) which allow the pilot to influence its movement (as described by variables such as pitch, roll or yaw). Although an aircraft is a highly coupled multivariable system, for several purposes its dynamics may be decoupled in two main blocks, corresponding to longitudinal and lateral movements.

Many problems in aircraft flight control may be adequately formulated in an optimal control framework. Some refer to linear dynamics and involve a quadratic cost. As an example, consider the problem of designing an auto-pilot for the longitudinal movement of an airplane, with the objective of maintaining small vertical acceleration. Similar examples for lateral auto-pilots could be provided as well.

The longitudinal perturbations of an airplane in horizontal cruising flight may approximately be described by the second order linear state space model

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -1/\tau & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ -Q_e \end{bmatrix} \delta$$

where α (perturbation from cruise angle of attack) and q (rate of perturbation from cruise pitch angle of the zero lift axis) are the state variables (assumed available for direct measurement) and δ (elevator deflection increment) is the manipulator variable. The parameters τ (lifting time constant), ω_0 (undamped pitch natural frequency) and Q_e (elevator effectiveness) depend on the airplane.

For such a system, a linear controller computes the manipulated variable from the state measurement by

$$\delta = -k_1\alpha - k_2 q$$

One possibility for computing the regulator gains k_1 and k_2 consists in selecting them such as to minimize

$$J = \lim_{T-t_0\to\infty} \int_{t_0}^{T} \left[\frac{\alpha^2(t)}{\alpha_0^2} + \frac{q^2(t)}{q_0^2} + \frac{\delta^2(t)}{\delta_0^2} \right] dt$$

where α_0, q_0 and δ_0 are constants. These constants play the role of weights of the terms forming the functional J and provide the design engineer with “knobs” for adjusting the closed loop response obtained.

Define $x := [\alpha \quad q]'$, $u = \delta$ and the matrices

$$Q := \begin{bmatrix} 1/\alpha_0^2 & 0 \\ 0 & 1/q_0^2 \end{bmatrix} \qquad R := \frac{1}{\delta_0^2}$$

With these definitions, the cost functional J is written as

$$J = \lim_{T-t_0\to\infty} \int_{t_0}^{T} [x'(t)Qx(t) + u'(t)Ru(t)] dt$$

which is a standard form to be considered below.

Other examples, *e. g.* in wide apart fields such as process control, ship auto-pilots or water management could also be presented, showing that a wealth of problems may be recast as optimal control problems formulated for linear plants and in which a quadratic cost is to be minimized – the LQ problem.

Although it can be argued that many “real life” problems are non-linear, LQ optimal control methods can contribute to the solution even in these cases. Indeed, there are classes of non-linear problems whose solution can be approximated by using LQ methods. For instance, algorithms for non-linear optimal design based on the theory of the second variation and quasi-linearization resort to the solution of a sequence of linear problems. In this vein, another example is provided by piecewise linear quadratic optimal control. This last strategy applies to a class of non-linear systems which can be approximated by piecewise linear models (local models). For each linear local model an LQ controller is designed, the global controller being obtained by a suitable concatenation of these local controllers.

From another point of view, the LQ design method is important because it yields controllers with a number of desirable properties. In particular, under mild assumptions, LQ controllers stabilize the closed-loop. Furthermore, if the state is available for direct measurement, the controller presents good gain and phase margins.

It should also be mentioned that basic LQ theory provides one of the key stones which supports the important class of design methods known as Predictive Control for linear plants (see the topic *Model-based Predictive Control*).

This article is concerned with the basic theory of the LQ regulator, in both continuous and discrete time, when the state of the plant to be controlled is available for direct measurement. A thorough understanding of the basic regulator is quite important for dwelling into its extensions, such as the situation in which the state is not available for direct measurement and must be estimated from input/output data (see *LQ-stochastic control*) or the tracking of nonzero references (see also *Servo control design*).

The approach followed here relies on the application of either Pontryagin's Maximum Principle (see *Pontryagin's Maximum Principle*) or Bellman's Dynamic Programming (see *Dynamic Programming*) to the type of dynamic model (linear) and cost function (quadratic) considered. In continuous time the plant to control is assumed to be described by a linear controllable state space model (see *System Characteristics: Stability, Controllability, Observability*), with the state available for direct measurement. The resulting controller for the regulation problem consists in a state feedback whose gain K depends on the solution P of a non-linear matrix differential equation, called Riccati equation. For finite optimization intervals, the matrix P, and therefore the vector of feedback gains K, is time dependent. When the optimization horizon grows unbound, however, the solution of the Riccati differential equation tends to a constant. The solution to the steady state LQ problem is thus given by a constant gain feedback control law, whose vector of gains depends on plant parameters, and a constant matrix P. This constant matrix satisfies the Riccati algebraic equation which results from equating to zero the time derivatives of P in the Riccati differential equations. Thus, the steady-state LQ controller is a specific type of state space controller (see *Design of State Space Controllers (Pole Placement) for SISO Systems*). Under mild mathematical conditions (*e. g.* controllability of the state space realization considered), which amount to the problem being well posed in engineering terms (meaning that the right actuators and sensors are being used), the closed loop poles are such that the closed loop system is stable and possesses good stability margins. For discrete time plants the results parallel these.

The structure of this paper is as follows: After this introduction (section 1), which provides motivation and an overall view, the LQ regulator problem in continuous time is considered in section 2 for a finite optimization interval. Section 3 presents the resulting regulator in steady state, yielded by the limit when the optimization horizon is made larger and larger, the corresponding properties being considered in section 4. Section 5 parallels the theory presented in sections 2 to 4 for discrete time problems. Section 6 briefly considers numerical methods for the LQ problem. Finally, section 7 draws conclusions.

2. The LQ Regulator in Continuous Time

Consider the linear time varying plant described by the state-space model

$$\dot{x}(t) = A(t)x(x) + B(t)u(t) \tag{1}$$

with initial condition $x(t_0) = x_0$ and where, for each $t \in [t_0, T]$, $x(t) \in \Re^n$, $u(t) \in \Re^m$, and the matrices $A(t)$ and $B(t)$ have compatible dimensions.

Associate to this plant the quadratic performance index

$$J(t_0) = \frac{1}{2}x'(T)S_T x(T) + \frac{1}{2}\int_{t_0}^{T}\left[x'(t)Q(t)x(t) + u'(t)R(t)u(t)\right]dt \tag{2}$$

where $S_T = S_T' \geq 0$ and, for each t, $Q(t) = Q(t)' \geq 0$ and $R(t) = R(t)' > 0$. Assume that A, B, Q and R have entries that are continuous functions of t. Furthermore, assume that the final time T is fixed and known and that no function of the final state is specified.

In order to determine the control $u_{\text{opt}}(t)$ for $t \in [t_0, T]$ that minimizes J, Pontryagin's Maximum Principle is applied to yield sufficient conditions on the control function maximizing $-J(t_0)$ (which is the same as minimizing $J(t_0)$). For this sake, write the Hamiltonian as

$$H(t) = \lambda'(Ax + Bu) + \frac{1}{2}(x'Qx + u'Ru) \tag{3}$$

where $\lambda(t) \in \Re^n$ is the co-state, which verifies the adjoint equation

$$-\dot{\lambda}' = \frac{\partial H}{\partial x} = x'Q + \lambda' A \tag{4}$$

together with the terminal condition

$$\lambda(T) = P(T)x(T) \tag{5}$$

The stationarity condition is that, along an optimal trajectory

$$\frac{\partial H}{\partial u} = \lambda' B + u' R = 0 \tag{6}$$

In terms of the co-state, the optimal control verifies thus

$$u(t) = -R^{-1} B' \lambda(t) \tag{7}$$

When the optimal control is used (*i. e.*, along an optimal trajectory), the state x and the co-state λ verify the system obtained by combining (1) and (4) with (7):

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B' \\ -Q & -A' \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{8}$$

The coefficient matrix in (8) is called the continuous Hamiltonian matrix.

Part of the unknown in (8) (the vector x) is specified for $t = t_0$, whereas the other part (the vector λ) is specified at the opposite end of the optimization interval, for $t = T$. For solving this two-point boundary value problem, assume that there is a matrix $P(t), t \in [t_0, T]$, such that, for each t

$$\lambda(t) = S(t)x(t) \tag{9}$$

In order to find the matrix $S(t)$, differentiate (9) and use (8) and again (9) to get

$$(\dot{S} + A'S + SA - SBR^{-1}B'S + Q)x = 0 \tag{10}$$

Since (9) is to hold for all x, the matrix S must be selected such as to verify the matrix Riccati equation for $t \le T$:

$$-\dot{S} = A'S + SA - SBR^{-1}B'S + Q \tag{11}$$

with terminal condition

$$S(T) = S_{\mathrm{T}} \tag{12}$$

In terms of the solution of the Riccati equation (11), the optimal controller is thus defined by the time varying state feedback control law

$$u_{\mathrm{opt}}(t) = -K_{\mathrm{opt}}(t)x(t) \qquad \text{cc} \tag{13}$$

with the optimal feedback gain given by

$$K_{\text{opt}}(t) = R^{-1}B'S(t) \tag{14}$$

The following theorem holds:

Theorem 2.1: LQ regulator in continuous time

The problem of minimizing the cost functional (2) when the plant state obeys model (1) and the matrices in (1) and (2) satisfy the assumptions made, is solved by the state feedback control law (13) with the optimal gain given by (14), where $S(t)$ is the matrix function staisfying the Riccati equation (11) with terminal condition (12). The matrix $S(t)$ exists for all $t \in [t_0, T]$.

Under the optimal control, the value of the performance index is given by

$$J(t_0) = \frac{1}{2}x'(t_0)S(t_0)x(t_0) \tag{15}$$

If $R(t) > 0$ for all $t \in [t_0, T]$ this actually corresponds to a minimum.

Solving the LQ regulator problem implies the solution of the Riccati equation (11) together with its terminal condition (12). This can be done off-line, the resulting function S being used to compute K_{opt} which is then stored in the memory of the control computer. During plant operation the control is then simply computed from (13) by retrieving $K_{\text{opt}}(t)$ from memory.

Example 2.1

Consider the open-loop unstable first-order linear system described by

$$\dot{x}(t) = x(t) + u(t) \tag{16}$$

with initial condition

$$x(0) = 1$$

x being a scalar. The control law is to be chosen such as to minimize the quadratic cost

$$J = \frac{1}{2}\int_0^T \left[x^2(t) + ru^2(t)\right]dt \tag{17}$$

Both $T > 0$ and $r > 0$ are fixed in each case, but several situations will be considered.

The solution to this LQ optimal control problem is given by

$$u(t) = -K(t)x(t) \tag{18}$$

where the optimal feedback gain is given by

$$K(t) = \frac{1}{r}S(t) \tag{19}$$

and S is a scalar function satisfying the first order Riccati differental equation

$$\dot{S}(t) = -2S(t) + \frac{1}{r}S^2(t) - 1 \tag{20}$$

with the terminal condition

$$S(T) = 0 \tag{21}$$

Eq. (20) may be solved by separation of variables. Alternatively, (20, 21) may be converted to an initial value problem by reversing time according to the change of variable $\tau = T - t$. In the transformed time scale τ the terminal condition (21) becomes an initial condition and standard numerical packages may then be used.

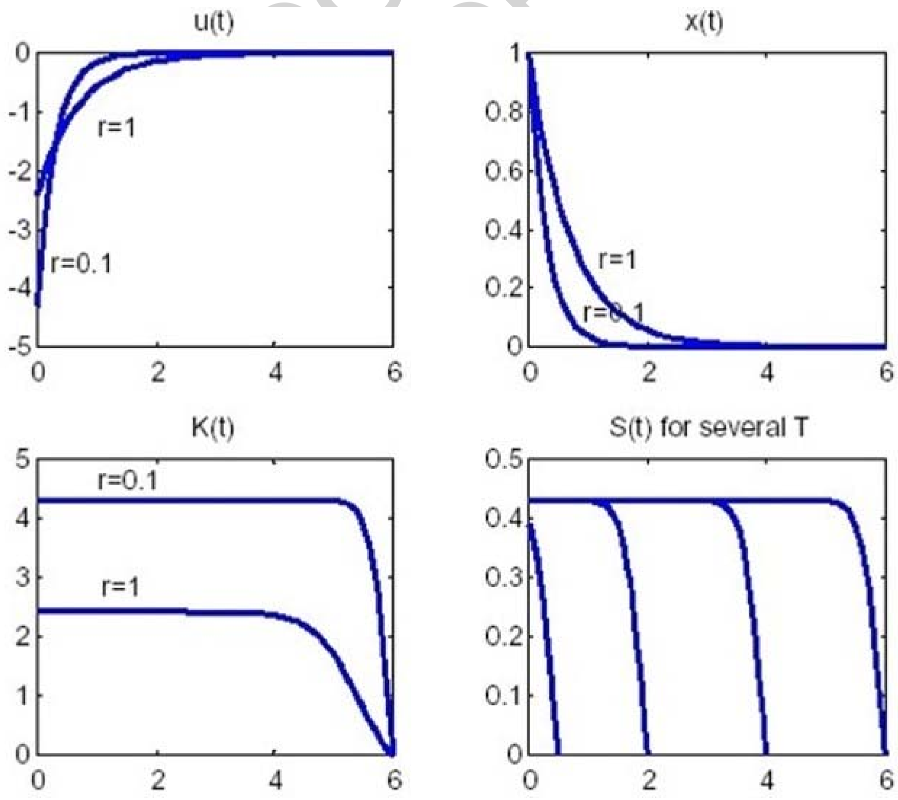

Figure 1 – Example 2.1: LQ control of a first order plant.

The numerical solution of the above problem may be seen in Figure 1 for two different values of r (namely, $r=1$ and $r=0.1$) and $T=6$. The graphic for S also shows, superimposed, the solution for different values of T and $r=0.1$. As can be seen, the controller gain K is almost constant during most of the optimization interval and then has a transient, for t close to T. Since K follows S, this is a consequence of the evolution of this function, the transient appearing in order to meet its specified terminal value.

In the period of time in which K is constant, its value is higher the smaller the value of r. This should be expected since r weights the manipulated variable energy, in the cost function J defined by (17). A smaller value of r leads to a more "energetic" control action, which will bring the state to zero faster. This is apparent in Figure 1, where $|u(0)|$ is higher for $r=0.1$ than for $r=1$, while the state function for $r=0.1$ is below the corresponding curve for $r=1$.

Look now at the curves for S obtained for different values of T (*viz.* $T=0.5$, 2, 4 and 6) and shown in the lower right graphic of Figure 1. When T is taken larger and larger, the period of time in which S is approximately constant increases more and more. In case T would be very large, one would expect the solution of the Riccati equation (and therefore the feedback gain) to be constant most of the time. Thus, the optimal performance would not be much different if the differential equation (20) is replaced by the following algebraic equation

$$-2S+\frac{1}{r}S^2-1=0 \tag{22}$$

Indeed, assuming S to be constant, its derivative vanishes and (20) reduces to (22). This motivates the consideration of a suboptimal strategy, known as the steady-state LQ controller, obtained, for time invariant plants and constant weight matrices, by letting $T-t_0\to\infty$.

-
-
-

Bibliography

Aldeen, M. and F. Crusca (1995). Multimachine power system stabiliser design based on new LQR approach. *IEE Proc. – Gener. Transm. Distrib.*, 142(5):494-502. [A systematic procedure based on LQ regulator theory for the design of excitation based stabilizers for large-scale interconnected power systems].

Anderson, B. D. O. and J. B. Moore (1989). *Optimal Control – Linear Quadratic Methods.* London: Prentice-Hall International. [A classical reference devoted to the design and properties of LQ controllers. Covers almost exclusively continuous time problems].

Arnold III, W. F. and A. J. Laub (1984). Generalized Eigenproblem Algorithms and Software for Algebraic Riccati Equations. *Proc. IEEE*, 72(12): 1746-1754. [Algorithms for the solution of the discrete time ARE based on the solution of a generalised eigenvalue problem].

Bender, D. J. and A. J. Laub (1987). The linear-quadratic optimal regulator for descriptor systems. *IEEE Trans. Autom. Control*, AC-32:672-688.

Bittanti, S., A. Laub and J. C. Willems (1991). *The Riccati equation*. Springer-verlab, Berlin. [A selection of papers covering topics on the Riccati equation].

Bryson, Jr., A. E. and Y.-C. Ho (1975). *Applied Optimal Control – Optimization, Estimation and Control.* Revised printing. New York, Washington, Philadelphia, London: Hemisphere Publishing Corporation (Taylor & Francis Group). [A classical reference on the application of Optimal Control to engineering problems. Covers the LQ problem for continuous time plants.].

Delfour, M. C. (1984). Linear Optimal control of systems with state and control variable delays. *Automatica*, 20 (1):69-77. [The extension of LQ regulator theory to systems with delay].

Francis, B. A. (1979). The Optimal Linear-Quadratic Time-Invariant Regulator with Cheap Control. *IEEE Trans. Autom. Control*, AC-24(4):616-621. [Studies the qualitative behavior of the optimal state and control trajectories when the infinite-time LQ regulator is considered as the weighting on the control energy tends to zero].

Fukuda, S. (1997). LQ control of sinusoidal current PWM rectifiers. *IEE Proc. – Electr. Power Appl.* 144(2):95-100. [An application of LQ control to a voltage source converter].

Gangsaas, D., K. Bruce, J. Blight and U. Ly (1986). Application of modern synthesis to aircraft control: Three case studies. *IEEE Trans. Autom. Control*, AC-31(11): 995-1014. [Early case studies of application of LQ and LQG to airplane control design].

Harvey, C. A. and G. Stein (1978). Quadratic weights for asymptotic regulator properties. *IEEE Trans. Autom. Control*, AC-23(3):378-387. [A procedure for selecting wight matrices in LQ optimal control, design is presented. The procedure is based on asymptotic modal characteristics of multivariable LQ-regulators as control weights tend to zero].

Kalman, R. E. (1969). Contributions to the Theory of Optimal Control. *Bol. Soc. Matem. Mex.*, 102-119. [An early account of the LQ problem].

Kalman, R. E. (1964). When is a linear control system optimal? *Trans. ASME, ser. D: J. Basic Eng.*, 86: 1-10. [The formulation of the inverse optimal control problem].

Kleinman, D. L. (1968). On an Iterative Technique for Riccati Equation Computations. *IEEE Trans. Autom. Control* AC-13(1):114-115. [The original presentation of Kleinman's algorithm for solving the steady state ARE].

Kreindler, E. (1968). Closed-loop sensitivity reduction of linear optimal control systems. *IEEE Trans. Autom. Control*, AC-13 (3):254-262. [An early paper tackling the sensitivity properties of LQ control].

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems*. New York, London, Sydney, Toronto: Wiley-Interscience. [A classical textbook. Presents the basic LQ theory based on state-space plant models for both the regulation and the servo problems].

Laub, A. (1979). A Schur method for solving algebraic Riccati equations. *IEEE Trans. Autom. Control*, AC-24 (6):913-921. [An algorithm based on the classical eigenvector approach for solving algebraic Riccati equations in both continuous and discrete time].

Lewis, F. L. and V. L. Syrmos (1995). *Optimal Control*. 2nd ed., New York, Chichester, Brisbane, Toronto, Singapore: John Wiley & Sons, Inc. (a Wiley-Interscience Publication). [A classical textbook on Optimal Control aiming mainly at engineering students. Covers the LQ problem for discrete and continuous time.]

MacLeod, C. and R. M. Goodall (1996). Frequency-shapping LQ control of Maglev suspension systems for optimal performance with deterministic and stochastic inputs. [A case study on the application of LQ control for frequency-shapping to an electromagnetic suspension].

Mäkilä, P. M. and H. T. Toivonen (1987). Computational Methods for Parametric LQ Problems – A survey. *IEEE Trans. Autom. Control*, AC-32(8):658-671. [A comprhensive survey of methods for solving parametric LQ problerms.].

Medanic, J., H. S. Tharp and W. R. Perkins (1988). Pole placement by performance criterion modification. *IEEE Trans. Autom. Control*, 33(5):469-472. [Invariance results for the algebraic Riccati equation enable the state weighting matrix and the degree of relative stability to be selected to position individually and arbitrarily the real parts of the eigenvalues of the optimal LQ system].

Mendel, J. M. and D. L. Gieseking (1971). Bibliography on the Linear-Quadratic-Gaussian Problem. *IEEE Trans. Autom. Control,* AC-16(6):847-869. [An extensive list of references up to the date of publication, covering as a main category the deterministic LQ problem.].

Mosca, E. (1995). *Optimal, Predictive, and Adaptive Control*. Englewood Cliffs, New Jersey: Prentice Hall Inc. [Includes a fairly complete exposition of LQ control of discrete plants].

Pappas, T., A. Laub and N. Sandell (1980). On the numerical solution of the discrete-time algebraic Riccati equation. *IEEE Trans. Autom. Control,* AC-25 (4):631-641. [Presents two algorithms for solution of the discrete time ARE based on the solution of a generalised eigenvalue problem].

Pindyck, R. S. (1973). *Optimal Planning for Economic Stabilization*. Amstredam, Oxford: North-Holland Publishing Company. [Provides a case study on the application of LQ control for deriving economic policies. A reduced complexity, linear model of post-Korean War U. S. economy is used to demonstrate and test the optimal control results.].

Rantzer, A. and M. Johansson (2000). Piecewise Linear Quadratic Optimal Control. *IEEE Trans. Aut. Control* 45(4):629-637. [The LQ problem is extended to piecewise linear systems through the use of piecewise quadratic cost functions].

Biographical Sketch

J. M. Lemos is professor of Automatic Control at IST, the engineering faculty of the Technical University of Lisbon, Portugal, where he is currently the Coordinator of the Post-Graduation Program of the Electrical Engineering Department. His research interests encompass computer control, adaptive control, control based on multiple models and modeling and control of industrial continuous processes. He got the Ph. D. degree at IST in 1989 after extensive periods of work at the University of Florence, Italy. He published over 100 research papers in journals, peer reviewed symposia and as book chapters.