CONTROLLER DESIGN USING POLYNOMIAL MATRIX DESCRIPTION

Didier Henrion
Laboratoire d'Analyse et d'Architecture des Systèmes, *Centre National de la Recherche Scientifique*, *Toulouse*, *France*.

Michael Šebek
Center for Applied Cybernetics, *Faculty of Electrical Engineering*, *Czech Technical University in Prague*, *Czech Republic*.

Keywords: polynomial methods, polynomial matrices, dynamics assignment, deadbeat regulation, H_2 optimal control, polynomial Diophantine Equation, polynomial spectral factorization, Polynomial Toolbox for Matlab

Contents

Summary

Polynomial matrix techniques can be used as an alternative to state-space techniques when designing controllers for linear systems. In this article, we show how polynomial techniques can be invoked to solve three classical control problems: dynamics assignment, deadbeat control and H_2 optimal control. We assume that the control problems are formulated in the state-space setting, and we show how to solve them in the polynomial setting, thus illustrating the linkage existing between the two approaches. Finally, we mention the numerical methods available to solve problems involving polynomial matrices.

1. Introduction

Polynomial matrices arise naturally when modeling physical systems. For example, many dynamical systems in mechanics, acoustics or linear stability of flows in fluid dynamics can be represented by a second order vector differential equation

$$A_0 x(t) + A_1 \dot{x}(t) + A_2 \ddot{x}(t) = 0. \tag{1}$$

Upon application of the Laplace transform, studying the above equation amounts to studying the characteristic matrix polynomial

$$A(s) = A_0 + sA_1 + s^2 A_2. \tag{2}$$

The constant coefficient matrices A_0, A_1 and A_2 are known as the stiffness, damping and inertia matrices, respectively, usually having some special structure depending on the type of loads acting on the system. For example, when A_0 is symmetric negative definite, A_1 is anti-symmetric and A_2 is symmetric positive definite, then the equation models the so-called gyroscopic systems. In the same way, third degree polynomial matrices arise in aero-acoustics.

In fluid mechanics, the study of the spatial stability of the Orr-Sommerfeld equation yields a quartic matrix polynomial. It is therefore not surprising to learn that most of the control design problems boil down to solving mathematical equations involving polynomial matrices. For historical reasons, prior to the sixties most of the control problems were formulated for scalar plants, and they involved manipulations on scalar polynomials and scalar rational functions.

The extension of these methods to the multivariable (multi-input, multi-output) case was not obvious at this time, and it has been achieved only with the newly developed concept of state-space setting. In the seventies, several multivariable results were therefore not available in the polynomial setting, which somehow renewed the interest in this approach. Now most of the results are available both in the state-space and in polynomial settings. In this article, we study in detail three standard control problems and their solution by means of polynomial methods.

The first problem is known as the *dynamics assignment* problem, a generalization of eigenvalue placement. The second problem is called the *deadbeat regulation* problem. It consists of finding a control law such that the state of a discrete-time system is steered to the origin as fast as possible, i.e. in a minimal number of steps.

The third and last problem is H_2 *optimal control*, where a stabilizing control law is sought that minimizes the H_2 norm of some transfer function. All these problems are formulated in the state-space setting, and then solved with the help of polynomial methods, to better illustrate the linkage between the two approaches. After describing these problems, we focus more on practical aspects, enumerating the numerical methods that are available to solve the polynomial equations.

2. Polynomial Approach To Three Classical Control Problems

2.1. Dynamics Assignment

We consider a discrete-time linear system

$$x_{k+1} = Ax_k + Bu_k \tag{3}$$

with n states and m inputs, and we study the effects of the linear static state feedback

$$u_k = -Kx_k \tag{4}$$

on the system dynamics. One of the simplest problem we can think of is that of enforcing closed-loop eigenvalues, i.e. finding a matrix K such that the closed-loop system matrix $A - BK$ has prescribed eigenvalues. In the polynomial setting, assigning the closed-loop eigenvalues amounts to assigning the characteristic polynomial det $(zI - A + BK)$. This is possible if and only if (A, B) is a reachable pair. (See *Pole Placement Control* for more information on pole assignment, and *System Characteristics: Stability, Controllability, Observability* for more information on reachability).

A more involved problem is that of enforcing not only eigenvalues, but also the eigenstructure of the closed-loop, system matrix. In the polynomial setting, the eigenstructure is captured by the so-called *similarity invariants* of matrix $A - BK$, i.e. the polynomials that appear in the Smith diagonal form of the polynomial matrix $zI - A + BK$. Rosenbrock's fundamental theorem captures the degrees of freedom one has in enforcing these invariants.

Let (A, B) be a reachable pair with reachability indices $k_1 \geq \cdots \geq k_m$. Let $c_1(z)$, ..., $cm(z)$ be monic polynomials such that $c_{i+1}(z)$ divides $c_i(z)$ and $\Sigma_{i=1}^{m} \deg c_i(z) = n$. Note that some of the $c_i(z)$ may be units. Then there exists a feedback matrix K such that closed-loop matrix $A - BK$ has similarity invariants $c_i(z)$ if and only if

$$\sum_{i=1}^{k} \deg c_i(z) \geq \sum_{i=1}^{k} k_i, \quad k = 1, ..., m. \tag{5}$$

The above theorem basically says that one can place the eigenvalues at arbitrary specified locations but the structure of each multiple eigenvalue is limited: one cannot split it into as many repeated eigenvalues as one might wish. Rosenbrock's result is constructive, and we now describe a procedure to assign a set of invariant polynomials by static state feedback. First we must find right coprime polynomial matrices $D_R(z)$ and $N_R(z)$ with $D_R(z)$ column-reduced and column-degree ordered, such that

$$(zI - A)^{-1} B = N_R(z) D_R^{-1}(z). \tag{6}$$

(See *Polynomial and Matrix Fraction Description* for the "definition" of a column-reduced matrix.) A column-reduced matrix can be put into column-degree ordered form by suitable column permutations. Then, we must form a column-reduced polynomial matrix $C(z)$ with invariant polynomials $c_1(z)$, ..., $c_p(z)$ which has the same column degrees as $D_R(z)$. Finally, we solve the equation

$$X_L D_R(z) + Y_L N_R(z) = C(z) \tag{7}$$

for constant matrices X_L and Y_L, and let

$$K = X_L^{-1} Y_L .$$

The above equation over polynomial matrices is called a *Diophantine polynomial equation*. Under the assumptions of Rosenbrock's theorem, there always exists a constant solution X and Y to this equation. As an example, take

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \tag{8}$$

as a reachable pair. We seek a feedback matrix K such that $A - BK$ has similarity invariants

$$c_1(z) = z^3 - z^2, \; c_2(z) = z. \tag{9}$$

Following a procedure of conversion from state-space form to matrix fraction description (MFD) form (see *Polynomial and Matrix Fraction Description*) we obtain

$$D_R(z) = \begin{bmatrix} z^2 & z \\ 0 & z^2 - z - 1 \end{bmatrix}, \qquad N_R(z) = \begin{bmatrix} 0 & z \\ 1 & 1 \\ z & z \\ 0 & 1 \end{bmatrix} \tag{10}$$

as a right coprime pair satisfying relation Eq. (6). The reachability indices of (A, B) are the column degrees of column-reduced matrix $D_R(z)$, namely $k_1 = 2$ and $k_2 = 2$. Rosenbrock's inequalities are satisfied since

$$\deg c_1(z) \geq 2, \; \deg c_1(z) + \deg c_2(z) \geq 4. \tag{11}$$

A column-reduced matrix with column degrees k_1, k_2 and $c_1(z)$, $c_2(z)$ as invariant polynomials is found to be

$$C(z) = \begin{bmatrix} z^2 & 0 \\ z & z^2 - z \end{bmatrix}. \tag{12}$$

Then, we have to solve the Diophantine equation

$$X_L \begin{bmatrix} z^2 & z \\ 0 & z^2 - z - 1 \end{bmatrix} + Y_L \begin{bmatrix} 0 & z \\ 1 & 1 \\ z & z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z^2 & 0 \\ z & z^2 - z \end{bmatrix}. \tag{13}$$

We find the constant solution

$$X_L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; Y_L = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix} \tag{14}$$

corresponding to the feedback matrix

$$K = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix} \tag{15}$$

yielding the desired dynamics.

-
-
-

Bibliography

Grimble M. J. and Kučera V. (Editors) (1996). *Polynomial Methods for Control Systems Design*, 255 pp. London: Springer-Verlag. [Collection of papers on the polynomial approach to solving both H_2 and H_∞ control and filtering problems.]

Kučera V. (1979). *Discrete Linear Control: The Polynomial Equation Approach*, 206 pp. Chichester, England: Wiley. [This is a pioneering textbook that paved the way for the use of polynomials, polynomial matrices and polynomial equations to solve linear control problems.]

Kučera V. (1991). *Analysis and Design of Discrete Linear Control Systems*, 472 pp. Prague: Academia. [Textbook that investigates connections between the state-space approach and the polynomial approach to several standard linear control problems. Most of the material proposed in this article was taken from this reference.]

Kwakernaak H. (1993). Robust Control and H_∞ Optimization-Tutorial Paper. *Automatica*, **29**(2), 255-273. [Tutorial exposition of the polynomial approach to H_∞ optimal regulation theory, with a focus on the mixed sensitivity problem for robust control system design.]

Kwakernaak H. and Šebek M. (1994). Polynomial J-Spectral Factorization. *IEEE Trans. Autom. Control*, 39(2), 315-328. [Collection of several algorithms for the *J*-spectral factorization of a polynomial matrix.]

Kwakernaak H. and Šebek M. (1998)). *The Polynomial Toolbox for Matlab*. Prague, CZ: PolyX Ltd., Web site at www.polyx.com or www.polyx.cz. [The Matlab Toolbox for polynomials, polynomial matrices and their application in systems, signals and control. Contains over 200 functions including various solvers for linear and quadratic matrix polynomial equations.]

Biographical Sketches

Didier Henrion was born in Creutzwald, Northeastern France, in 1971. He received the "Diplôme d'Ingénieur" (Engineer's Degree) and the "Diplôme d'Etudes Approfondies" (Masters' Degree) with specialization in control from the Institut National des Sciences Appliquées (INSA, National Institute for

Applied Sciences), Toulouse, Southwestern France, in September 1994. From October 1994 to December 1995, he was a research assistant at the Centro de Automatización Industrial of the Universidad Simón Bolívar, Caracas, Venezuela. From February 1996 to December 1998 he was a Ph.D. student at the Ustav Teorie Informace a Automatizace (UTIA, Institute of Information Theory and Automation), Prague, Czech Republic.

He received his "C.Sc." (Ph.D.) degree from the Akademie Věd České Republiky (Academy of Sciences of the Czech Republic) in December 1998.

From October 1996 to August 1999 he was a Ph.D. student at the Laboratoire d'Analyse et d'Architecture des Systèmes (LAAS, Laboratory of Analysis and Architecture of Systems) of the Centre National de la Recherche Scientifique (CNRS, National Center for Scientific Research) in Toulouse, France. He received his "Diplôme de Doctorat" (Ph.D. degree) from INSA in October 1999. Since October 2000, he has been a "Chargé de Recherche au CNRS" (CNRS Researcher) at LAAS under the scientific responsibility of Jacques Bernussou. He is also holding a secondary appointment as a researcher at UTIA under the scientific responsibility of Michael Šebek. In October 2001 he was awarded a Young Researcher Grant by CNRS. Didier Henrion serves as an Associate Editor of the IFAC Automatica journal as well as an Associate Editor at the Conference Editorial Board of the IEEE Control Systems Society. He is vice-chair of the IFAC Technical Committee on Control Design. He is a regular member of the panel of reviewers of the American Mathematical Society and Zentralblatt Math.

Michael Šebek was born in Prague, Czechoslovakia, in 1954. He obtained his Ing. degree in electrical engineering with distinction from the Czech University of Technology, Prague in 1978, his CSc. degree in control engineering and his DrSc. degree in control theory from the Czech Academy of Sciences in 1981 and 1995, respectively. He held visiting positions at the Strathclyde University, Glasgow, UK; Universiteit Twente, NL; and at ETH Zurich, CH.

Michael Šebek is Head of Control Engineering Department and Manager of the Center of Applied Cybernetics at the Czech Technical University in Prague. He is also Head of Control Theory Department in the Institute of Information Theory and Automation of the Czech Academy of Sciences and Chief Executive Officer of PolyX, Ltd. Prague.

His research interests include linear systems and signals, robust control, algorithms and software for control and filter design. He published well over 160 research papers and contributed to 10 books. Science Citation Index lists 170 citations of his works.

Michael Šebek has led numerous international and national research projects. In particular, he has coordinated EUROPOLY, the European Network of Excellence for Industrial Applications of Polynomial Methods.

He co-founded PolyX Ltd., Prague (www.polyx.com), the company producing software for systems, signals and control based on polynomial methods. He is currently the CEO of PolyX and leads the development of the company main product: the Polynomial Toolbox for Matlab.

Michael Šebek served as Associate Editor of the European Journal of Control. Within IFAC, he is vice-chair of the Policy Committee and member of the Technical Committee on Control Design. He is a Senior Member of the IEEE serving in the Czechoslovakia Section Executive Committee. He was the founding chairman of the Czech CSS Chapter. He is also a member of AMS, NYAS and SIAM.

Michael Šebek holds the Czech Government Prize for his achievements in algebraic control theory. He is listed in Marquis “Who's Who in the World and in Who's Who in Science and Engineering”.

For more info, visit www.michaelŠebek.cz .