μ-SYNTHESIS

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Contents

Summary

This chapter presents a perspective on the design of robust controllers via μ-synthesis techniques. μ-synthesis is a multivariable feedback controller design process that accounts for robustness to structured uncertain variations in the open-loop plant dynamics during the synthesis process.

1. Introduction

μ-synthesis concerns the design of multivariable feedback controller that is robust to structured uncertain variations in the open-loop plant model. μ refers to the structured singular value which is the reciprocal of the multivariable stability margin, denoted K_m. The goal is to automate the synthesis of multivariable feedback controllers that achieve desired performance objectives and are insensitive to modeled plant uncertainty.

Stability margins for multivariable systems can be traced back to Sandberg and Zames in the 1960's. Their input-output results, which did not include model uncertainty, were based on conic-sectors, positivity and loop-gain. They form the basis of the structured singular value and multivariable gain margin. Their small-gain nonlinear stability results were in fact based on singular values which now play a significant role in multivariable system theory and analysis. The notation of the multivariable stability margin, K_m, was introduced by Safonov and Athans. The structured singular value terminology, μ, was introduced by Doyle. The calculation of these analysis metrics required solution of a scaled singular value problem. Optimizing the singular value of the matrix over a set of diagonal scaling reduced the conservatism of these metrics. Initial algorithms for optimally scaled singular value problems focused on complex valued uncertainty. Algorithms for optimal scaling techniques and their generalization to real valued uncertainty followed. Through out the remainder of this chapter, the concept of the structured singular value μ and the multivariable stability margin K_m, the reciprocal of μ, will be denoted as μ to improve the readability of the text.

The structured singular value is the appropriate tool for analyzing the robustness (both stability and performance) of a linear, time-invariant system with structured uncertainty. Hence, a multivariable controller synthesis technique which directly seeks to minimize μ would be advantageous. The concept of μ synthesis was introduced in the 1980's, combining H_∞ control design and the diagonal scaling techniques from the structured singular value. The structured uncertainty was modeled as H_∞-norm bounded uncertain gains in one or more system inputs and outputs. The algorithms, denoted as $D-K$ iteration, involves iteratively optimizing a set of diagonal scaling frequency response matrices $D(j\omega)$ for a fixed controller $K(s)$.

Each of these optimizations are known to be convex individually, though the combined problem is not convex. Thus the $D-K$ iteration algorithm for μ-synthesis cannot be guaranteed to be globally optimal. It practice, this algorithms works very well and are computational efficient and fast. Each $D-K$ iteration improves the bound on performance and robustness of the controller until subsequent iterations show no improvement. These algorithms are available in commercial software packages and have been applied successfully to numerous real applications over the past 15 years.

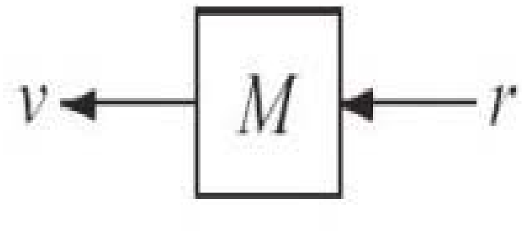
The $D-K$ iteration algorithm for complex model uncertainty, described in the previous paragraph, has been extended to include real parameter uncertainty. This algorithm is denoted as $(D,G)-K$ iteration and has been applied to several real applications. An alternative approach to μ-synthesis replaces the D-scale state-space realization step in the algorithm with optimization of a fixed order diagonal scaling matrix $D(s)$. This approach is also discussed in this section.

2. Control Design via $D-K$ Iteration

The structured singular value, μ, is a tool for analyzing the stability and performance robustness of a system subjected to structured, linear fractional perturbations. In this section, the mechanics of a controller design methodology based on structured singular value objectives are presented. These methods rely heavily on the upper bound for μ.

2.1. Linear Fractional Transformations, LFTs

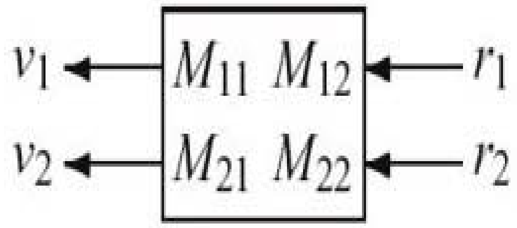
Linear Fractional Transformations (LFT) are a powerful and flexible approach to represent uncertainty in matrices and systems. Consider first a complex matrix M as in Figure 1, relating vectors r and v,



$$v = Mr$$

Figure 1: Matrix M relating r and y

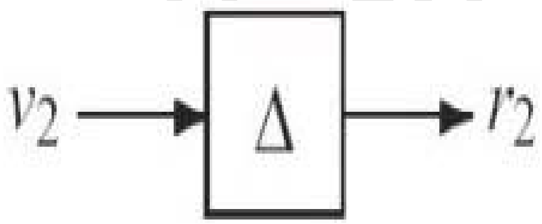
If r and v are partitioned into a top part and bottom part, then we can draw the relationship in more detail, explicitly showing the partitioned matrix M in Figure 2.



$$v_1 = M_{11}r_1 + M_{12}r_2$$
$$v_2 = M_{21}r_1 + M_{22}r_2$$

Figure 2: Partitioned matrix M relating r and y

Suppose a matrix Δ relates v_2 to r_2, Figure 3, as



$$r_2 = \Delta v_2$$

Figure 3: Matrix Δ relating v_2 and r_2

The linear fractional transformation of M by Δ interconnects these two elements, as shown in Figure 4.

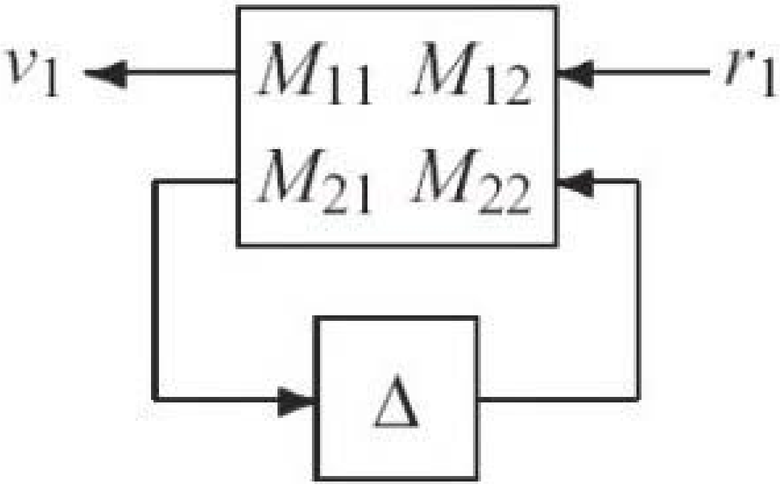
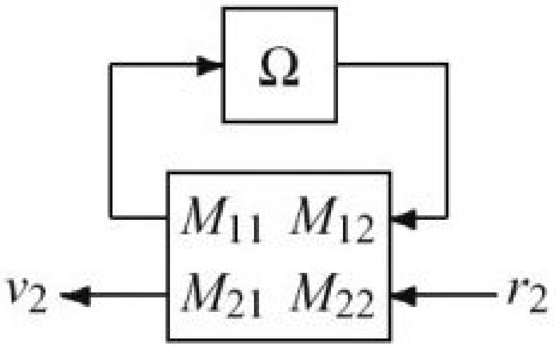


Figure 4: Lower linear fractional transformation of M and Δ

Eliminate v_2 and r_2, leaving the relationship between r_1 and v_1

$$\begin{aligned} v_1 &= \underbrace{\left[M_{11} + M_{12}\Delta\left(I - M_{22}\Delta\right)^{-1}M_{21}\right]}_{F_L(M,\Delta)} r_1 \\ &= F_L\left(M,\Delta\right) r_1 \end{aligned}$$

The notation F_L indicates that the *lower* loop of M is closed with Δ. If the *upper* loop of M is closed with Ω, then we have and upper linear fractional transformation as shown in Figure 5



$$v_2 = F_U\left(M,\Omega\right) r_2$$

Figure 5: Upper linear fractional transformation of M and Ω

where

$$F_U\left(M,\Omega\right) := \left[M_{22} + M_{21}\Omega\left(I - M_{11}\Omega\right)^{-1}M_{12}\right].$$

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Biographical Sketch

Gary J. Balas received the BS and MS degree in civil and electrical engineering from UC Irvine and the PhD degree in Aeronautics from the California Institute of Technology in 1990. He is a Professor in the Department of Aerospace Engineering and Mechanics at the University of Minnesota. He is a co-organizer and developer of the MUSYN Robust Control Short Course and the μ-Analysis and Synthesis Toolbox used with MATLAB and the president of MUSYN Inc. He is an Associate Fellow of the AIAA and a Senior Member of the IEEE.