CONTROL OF 2-D SYSTEMS

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Contents

Summary

Standard models of 2-D linear systems are presented, and relationships between the models are established. Solutions to the models and general response formulae are given. The realization problem is formulated and solved for standard 2-D models. Stability criteria are presented, and the eigenvalue assignment problem is formulated and solved. Necessary and sufficient conditions for controllability and observability are given.

1. Introduction

A growing interest in problems involving signals and systems that depend on more than one variable has developed over the past two decades. Two-dimensional (2-D) discrete systems are dynamical systems described by difference equations in two independent variables. The 2-D and the general case n-D (n>2) signals and system have been studied in relation to several modern engineering fields, such as multidimensional digital filtering, multivariable network realizability, multidimensional system synthesis, digital picture processing, seismic data processing, X-ray image enhancement, the enhancement and analysis of aerial photographs for detection of forest fires or crop damage, the analysis of satellite weather photos, image deblurring, etc.

The 2-D systems theory has also many industrial applications for example in industrial processes such as paper making, plastic film extrusion and steel sheet formation.

Most of the major results concerning the multidimensional signals and systems have been developed for 2-D cases using the models introduced by Roesser (1975) and by Fornasini and Marchesini (1976-1978). Recently a dynamical development has been observed for the repetitive systems theory that found wide acceptance for a variety of applications.

The behavioral approach to theoretical questions for 2-D and n-D basic systems has already been proven to be a very powerful tool for solving long standing open problems.

2. Standard Models of 2-D Linear Systems

Roughly speaking, two-dimensional (2-D) discrete systems are dynamic systems described by difference equations in two independent variables. The most popular standard 2-D model, introduced by Roesser (1975), has the form

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{i,j} \tag{1a}$$

$$y_{i,j} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + Du_{i,j}\ ,\ i,j \in Z_+ \tag{1b}$$

where $x^h_{ij} \in R^{n_1}$ is the horizontal state vector at the point $(i,j) \in Z_+ \times Z_+$,

$x^v_{ij} \in R^{n_2}$ is the vertical state vector at the point $(i,j) \in Z_+ \times Z_+$,

$u_{ij} \in R^m$ is the input vector,

$y_{ij} \in R^p$ is the output vector,

$A_{kl} \in R^{n_k \times n_l}$, $B_k \in R^{n_k \times m}$, $C_l \in R^{p \times n_l}$, $D \in R^{p \times m}, k,l = 1,2,$

$R^{n \times m}$ is the set of $n \times m$ real matrices and Z_+ is the set of nonnegative integers.

Boundary conditions for (1a) are given by

$$x^h_{0j},\ j \in Z_+, x^v_{i0},\ i \in Z_+ \tag{2}$$

The model (1) is called the standard 2-D Roesser model (RM). The standard first 2-D Fornasini-Marchesini model (FF-MM) is defined by the equations (1976).

$$\bar{x}_{i+1,j+1} = A_0\bar{x}_{ij} + A_1\bar{x}_{i+1,j} + A_2\bar{x}_{i,j+1} + Bu_{ij} \tag{3a}$$

$i, j \in Z_+$

$$y_{ij} = C\bar{x}_{ij} + Du_{ij} \tag{3b}$$

where $\bar{x}_{ij} \in R^n$ is the local state vector at the point $(i, j) \in Z_+ \times Z_+$,

$u_{ij} \in R^m$ is the input vector,

$y_{ij} \in R^p$ is the output vector,

$A_k \in R^{n\times n}$, $k = 0,1,2,$ $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$

Boundary conditions for (3a) are given by

$$x_{i0},\ i \in Z_+,\ x_{0j},\ j \in Z_+ \tag{4}$$

The standard second 2-D Fornasini-Marchesini model (SF-MM) is defined by the equations (1978).

$$x_{i+1,j+1} = A_1x_{i+1,j} + A_2x_{i,j+1} + B_1u_{i+1,j} + B_2u_{i,j+1} \tag{5a}$$

$$y_{ij} = Cx_{ij} + Du_{ij}\ ,\qquad i, j \in Z_+ \tag{5b}$$

where $x_{ij} \in R^n$ is the local state vector at the point $(i, j) \in Z_+ \times Z_+$,

$u_{ij} \in R^m$ is the input vector,

$y_{ij} \in R^p$ is the output vector,

$A_k \in R^{n\times n}$, $B_k \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$, $k = 1,2$

Boundary conditions for (5a) are given by

$$x_{i0},\ i = 1,2,...\ ,\ \ x_{0j},\ j = 1,2,... \tag{6}$$

For $A_0 = A_1A_2 = -A_2A_1$ from (3) we obtain the standard 2-D Attasi model (AM) of the form

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} - A_1 A_2 x_{ij} + Bu_{ij} \tag{7a}$$

$$y_{ij} = Cx_{ij} + Du_{ij} \qquad\qquad i, j \in Z_+ \tag{7b}$$

Boundary conditions for (7a) are given by (4).

3. Relationship between Models.

Defining

$$x_{ij}^h = \bar{x}_{i,j+1} - A_1 x_{ij}\,, x_{ij}^v = \bar{x}_{ij}$$

and using (3a) we may write

$$x_{i+1,j}^h = A_0 x_{ij}^v + A_2\left(x_{ij}^h + A_1 x_{ij}^v\right) + Bu_{ij} = A_2 x_{ij}^h + \left(A_0 + A_2 A_1\right)x_{ij}^v + Bu_{ij} \tag{8a}$$

and

$$x_{i,j+1}^v = x_{ij}^h + A_1\, x_{ij}^v \tag{8b}$$

From (8) we have

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_2 & A_0 + A_2 A_1 \\ I_n & A_1 \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_{ij} \qquad (I_n \text{ - the identity matrix})$$

$$y_{ij} = \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix}$$

Thus, FF-MM can be recast in RM with

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_2 & A_0 + A_2 A_1 \\ I_n & A_1 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} C_1 & C_2 \end{bmatrix} = [0 \;\; C], D = 0$$

If $A_{21} = I_n, B_2 = 0$ and $C_1 = 0$ then RM can also be recast in FF-MM with

$$A_0 = A_{12} - A_{11}A_{22}, A_1 = A_{22}, A_2 = A_{11}, B = B_1, C = C_2 \tag{10}$$

In particular case AM can be recast in RM with

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_2 & 0 \\ I_n & A_1 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix}, [C_1 \;\; C_2] = [0 \;\; C], D = 0 \tag{11}$$

and if $A_{12} = 0, A_{21} = I_n, A_{11}A_{22} = A_{22}A_{11}, B_2 = 0$ and $C_1 = 0$ then RM can also be reformulated in AM with

$$A_1 = A_{22}, A_2 = A_{11}, B = B_1, C = C_2 \tag{12}$$

Defining

$$x_{ij} = \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix}$$

we can write (1) in the form

$$x_{i+1,j+1} = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} x_{i+1,j} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} x_{i,j+1} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u_{i+1,j} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_{i,j+1}$$

and

$$y_{ij} = [C_1 \;\; C_2] x_{ij} + Du_{ij}$$

Thus, RM is a particular case of SF-MM with

$$A_1 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, A_2 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, B_2 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, C = [C_1 \;\; C_2] \tag{13}$$

FF-MM can be embedded in SF-MM.

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Bibliography

Bose N.K. (1986) *Multidimensional Systems, Progress, Direction and Open Problems*. Dordrecht, Holland. [This monograph summarizes the developments in 2-D systems theory and presents open problems].

Fornasini E., and Marchesini G. (1976) *State space realization of two-dimensional filters, IEEE Trans. Autom. Control* **AC-21**, 484-491. [The first Fornasini-Marchesini model was presented].

Fornasini E., and Marchesini G. (1978) Doubly indexed dynamical systems: State space models and structural properties. *Math. Syst. Theory* **12** [The second Fornasini-Marchesini model was presented].

Gałkowski K. (2001) *State-space Realizations of Linear Systems with Extensions to the General nD (n>2) Case*. Springer-Verlag, London. [A new elementary approach to state-space realization problem is presented].

Kaczorek T. (1993) *Linear Control Systems, vol. 2*. Research Studies Press and J. Wiley, New York. [This monograph contains new results in singular 2-D linear systems].

Kaczorek T. (1985) *Two-Dimensional Linear Systems*, Springer-Verlag, New York. [This monograph summarizes the classical 2-D systems theory].

Kaczorek T (2001) *Positive 1D and 2D Systems*. Springer-Verlag. [This monograph summarizes recent developments in 1D and 2D positive systems].

J.Klamka J. (1991) *Controllability of Dynamical Systems*, Kluwer Academic Publ. Dordrecht. [The monograph summarizes the recent results in controllability of 1-D and 2-D linear systems].

RoesserR.P. (1975) *A discrete state-space model for linear image processing, IEEE Trans. Autom.Control*, **AC-20**, 1-10. [The new 2-D Roesser model is introduced].

Rogers E., and Owens D.H. (1992) *Stability Analysis for Linear Repetitive Processes.* Springer-Verlag.[This monograph presents stability analysis of linear repetitive processes].

Biographical Sketch

Tadeusz Kaczorek received MSc., PhD and DSc degrees in Electrical Engineering from Warsaw University of Technology in 1956, 1962 and 1964, respectively. In the period from 1968-69, he was the Dean of Electrical Engineering Faculty and in the period 1970-73, he was the Pro-Rector of Warsaw University of Technology. Since 1971, he has been Professor and since 1974 full Professor at Warsaw University of Technology. In 1986, he was elected a member of Polish Academy of Sciences. During the period 1988-91, he was the director of the Research Centre of Polish Academy of Sciences in Rome. His research interests cover the theory of systems and automatic control systems theory, specially, singular multidimensional systems, positive multidimensional systems and singular positive 1D and 2D systems. He has published 18 books (5 in English) and over 600 scientific papers in journals like IEEE Transactions on Automatic Control, IEEE Transactions on Neural Networks, Multidimensional Systems and Signal Processing, International Journal of Control etc. He has presented more than 80 invited papers at international conferences and world congresses. He has given invited lectures in more than 50 universities in USA, Canada, UK, German, Italy, France, Japan, Greece, etc. He has been member of many international committees and editorial boards.