ANALYSIS OF NONLINEAR CONTROL SYSTEMS

Hassan K. Khalil
Department of Electrical and Computer Engineering, Michigan State University, USA.

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Summary

This article presents some useful tools for the analysis of nonlinear control systems. The article starts by reviewing some facts about the existence, uniqueness, and continuity of the solutions of nonlinear differential equations. Then, it presents sensitivity analysis, small-gain and passivity theorems for the stability of feedback connections, and the averaging and singular perturbation methods for the analysis of two-time-scale systems.

1. Introduction

When engineers analyze and design nonlinear control systems, they need a wide range of nonlinear analysis tools. In this article, we describe some tools which are not covered in other articles. We start by recalling some fundamental properties of ordinary differential equations like existence, uniqueness, and continuity of solutions. Then we derive sensitivity equations which describe the effect of small parameter variations on the performance of the system. Stability analysis plays a central role in control engineering. Lyapunov and input-output stability concepts are presented in other articles (see *Stability theory, Popov and circle criterion, Lyapunov stability and Input-output stability*). We do not repeat these concepts here but we consider stability of the feedback connection of Figure 1. If the feedback components H_1 and H_2 are stable in some sense, it is interesting to know under what conditions will the feedback connection be stable. Towards that end, we present the small-gain theorem and passivity theorems. Finally, we present the averaging and singular perturbation asymptotic methods, which reveal multiple-time-scale structures inherent in many dynamical systems. In such structures, some variables move in time faster than other variables, leading to the classification of variables as “slow” and “fast”.

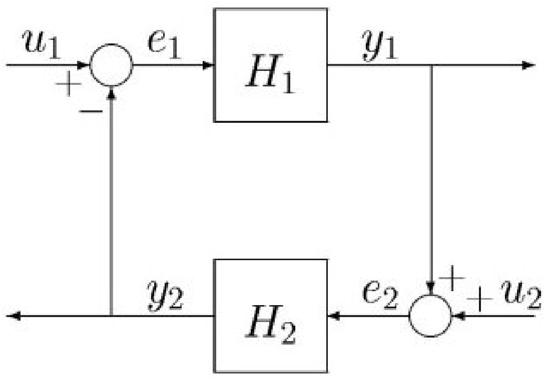

Figure 1: Feedback connection.

Throughout the article we deal with dynamical systems represented by the state model

$$\dot{x} = f(t,x,u), \quad y = h(t,x,u)$$

where $x, u,$ and y are vectors representing the state, input, and output, respectively, and $\dot{x}$ is the derivative of x with respect to the time variable t. For the state equation without input $\dot{x} = f(t,x)$, a point x^* is an equilibrium point if $f(t,x^*) = 0$ for all t; hence a solution starting at x^* stays at x^* for all time. Specializing further to the case $\dot{x} = f(x)$, we can see that equilibrium points are the real roots of $0 = f(x)$. To avoid duplication, we assume familiarity with the terminology and results of Lyapunov theory (see *Lyapunov stability*).

2. Fundamental Properties

Existence, uniqueness, and continuous dependence of solutions on initial conditions and parameters are fundamental properties for the state equation $\dot{x} = f(t,x)$ to be a useful mathematical model of a physical system. For the model to predict the future state of the system from its current state at t_0, the initial value problem

$$\dot{x} = f(t,x), \quad x(t_0) = x_0 \tag{1}$$

must have a unique solution. Local existence and uniqueness of solutions on some interval $[t_0, t_0 + \delta]$ are guaranteed if the function $f(t,x)$ is locally Lipschitz at (t_0, x_0); that is, if it satisfies the Lipschitz condition

$$\| f(t,x) - f(t,y) \| \le L \| x - y \| \tag{2}$$

for all (t,x) and (t,y) in some neighborhood of (t_0, x_0).

When $f(x)$ and x are scalars, the Lipschitz condition can be written as $|f(y) - f(x)| / |y - x| \le L$ which implies that the absolute value of the slope of $f(x)$ is less than L. Therefore, any function $f(x)$ that has infinite slope at some point is not locally Lipschitz at that point. For example, any discontinuous function is not locally Lipschitz at the point of discontinuity. As another example, the continuous function $f(x) = x^{1/3}$ is not locally Lipschitz at $x = 0$. It can be easily seen that the scalar equation $\dot{x} = x^{1/3}$ with $x(0) = 0$ does not have a unique solution, as both $x(t) = (2t/3)^{3/2}$ and $x(t) \equiv 0$ are valid solutions.

When $f(t,x)$ is continuous in t and has continuous first partial derivatives with respect to x in a domain D, it is locally Lipschitz in D. If D is the whole space and the partial derivatives $\partial f_i / \partial x_j$ are globally bounded, f will be globally Lipschitz; that is, the Lipschitz condition (2) is satisfied for all x and y with same constant L.

Local existence and uniqueness on an interval $[t_0, t_0 + \delta]$ can be extended to global existence on the interval $(t_0, \infty]$ if

- f is globally Lipschitz, or
- f is locally Lipschitz and it can be argued that the solution of $\dot{x} = f(t,x)$ cannot leave a compact set.

For example, in the scalar system $\dot{x} = -x^3$ the function f is not globally Lipschitz, but for any initial state $x(0) = a$ the solution cannot leave the compact set $\{|x| \le |a|\}$. Thus, the equation has a unique solution for all $t \ge 0$.

For the solution of the state equation (1) to be of any interest, it must depend continuously on the initial state x_0 and the right-hand side function $f(t,x)$. The Lipschitz condition guarantees such continuity properties. Let $f(t,x)$ be continuous in t and Lipschitz in x on $[t_0, t_1] \times D$, for some domain D, with a Lipschitz constant L. Let $y(t)$ and $z(t)$ be solutions of

$\dot{y} = f(t,y),\ y(t_0) = y_0$, and $\dot{z} = f(t,z) + g(t,z),\quad z(t_0) = z_0$

such that $y(t), z(t) \in D$ for all $t \in [t_0, t_1]$. Suppose $\|g(t,x)\| \le \mu$ for all $t \in [t_0, t_1]$, $x \in D$ for some $\mu > 0$. Then,

$$\|y(t) - z(t)\| \le \|y_0 - z_0\| \exp[L(t - t_0)] + \frac{\mu}{L}\{\exp[L(t-t_0)] - 1\}$$

$\forall t \in [t_0, t_1]$. This inequality shows that the solution of (1) depends continuously on x_0 and f. The inequality, however, assumes that both $y(t)$ and $z(t)$ are defined on the interval $[t_0, t_1]$. In fact, if one of the two solutions is defined, then by continuity the other solution should be defined as well. This can be shown if the perturbation of f is parameterized by a vector of constant parameters λ. Let $f(t,x,\lambda)$ be continuous in (t,x,λ) and locally Lipschitz in x (uniformly in t and λ) on $[t_0, t_1] \times D \times \{\|\lambda - \lambda_0\| \le c\}$, for some domain D. Let $y(t,\lambda_0)$ be a solution of $\dot{x} = f(t,x,\lambda_0)$ with $y(t_0,\lambda_0) = y_0 \in D$. Suppose $y(t,\lambda_0)$ is defined and belongs to D for all $t \in [t_0, t_1]$. Then, given $\varepsilon > 0$, there is $\delta > 0$, such that if $\|z_0 - y_0\| < \delta$ and $\|\lambda - \lambda_0\| < \delta$, then there is a unique solution $z(t,\lambda)$ of $\dot{x} = f(t,x,\lambda)$ defined on $[t_0, t_1]$ with $z(t_0,\lambda) = z_0$ such that $\|z(t,\lambda) - y(t,\lambda_0)\| < \varepsilon$ for all $t \in [t_0, t_1]$.

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Bibliography

Hale J.K. (1969). *Ordinary Differential Equations*. New York: Wiley-Interscience.

Khalil H.K. (2002). *Nonlinear Systems*. Upper Saddle River, NJ: Prentice Hall, third edn.

Kokotovic P.V., Khalil H.K., O' Reilly J. (1999). *Singular Perturbations Methods in Control*: *Analysis and Design.* Philadelphia: SIAM.

Miller R.K., Michel A.N. (1982). *Ordinary Differential Equations*. New York: Academic Press.

Sanders J.A., Verhulst F. (1985). *Averaging Methods in Nonlinear Dynamical Systems.* No.59 in Applied Mathematical Sciences, New York: Springer-Verlag.

Sastry S. (1999). *Nonlinear Systems*: *Analysis, Stability, and Control*. New York: Springer.

Sastry S., Bodson M. (1989). *Adaptive Control*. Englewood Cliffs, NJ: Prentice Hall.

Vidyasagar M. (1993). *Nonlinear Systems Analysis*. Englewood Cliffs, NJ: Prentice Hall, second edn.

Biographical Sketch

Hassan K. Khalil received the B.S. and M.S. degrees from Cairo University, Cairo, Egypt, and the Ph. D. degree from the University of Illinois, Urbana-Champaign, in 1973, 1975, and 1978, respectively, all in Electrical Engineering.

Since 1978, he has been with Michigan State University, East Lansing, where he is currently University Distinguished Professor of Electrical and Computer Engineering. He has consulted for General Motors and Delco Products.

He has published several papers on singular perturbation methods, decentralized control, robustness, nonlinear, control, and adaptive control. He is author of the book Nonlinear Systems (Macmillan, 1992; Prentice Hall, 1996 and 2002), coauthor, with P. Kokotovic and J.O'Reilly of the book Singular Perturbation Methods in Control: Analysis and Design (Academic Press, 1986; SIAM 1999), and coeditor, with P. Kokotovic, of the book Singular Perturbation in Systems and Control (IEEE Press, 1986). He was the recipient of the 1983 Michigan State University Teacher Scholar Award, the 1989 George S. Axelby Outstanding Paper Award of the IEEE Transactions on Automatic Control, the 1994 Michigan State University Withrow Distinguished Scholar Award, the 1995 Michigan State University Distinguished Faculty Award, the 2000 American Automatic Control Council Ragazzini Education Award, and the 2002 IFAC Control Engineering Textbook Prize. He is an IEEE Fellow since 1989.

Dr. Khalil served as Associate Editor of IEEE Transactions on Automatic Control, 1984-1985; Registration Chairman of the IEEE-CDC Conference, 1984; Finance Chairman of the 1987 American Control Conference (ACC); Program Chairman of the 1988 ACC; General Chair of the 1994 ACC; Associate Editor of *Automatica*, 1992-1999; Action Editor of Neural Networks, 1998-1999; and Member of the IEEE-CSS Board of Governors, 1999-2002. Since 1999, he has been serving as Editor of *Automatica* for nonlinear systems and control.