LYAPUNOV DESIGN

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Contents

Summary

This chapter gives an overview on some state-of-the-art approaches of Lyapunov design by dividing systems into several distinct classes, though in general there is no systematic procedure in choosing a suitable Lyapunov function candidate for controller design to guarantee the closed-loop stability for a given nonlinear system. After a brief introduction and historic review, this chapter sequentially presents (i) the basic concepts of Lyapunov stability and control Lyapunov functions, (ii) Lyapunov equations and model reference adaptive control based on Lyapunov design for matched systems, (iii) Lyapunov redesign, adaptive redesign and robust design for matched systems, (iv) adaptive backstepping design for unmatched nonlinear systems, (v) Lyapunov design by exploiting physical properties for special classes of systems, and (vi) design flexibilities

and considerations in actual design.

1. Introduction

Lyapunov design has been a primary tool for nonlinear control system design, stability and performance analysis since its introduction in 1892. The basic idea is to design a feedback control law that renders the derivative of a specified Lyapunov function candidate negative definite or negative semi-definite. Lyapunov's direct method is a mathematical interpretation of the physical property that if a system's total energy is dissipating, then the states of the system will ultimately reach an equilibrium point. The basic idea behind the method is that, if there exists a kind of continuous scalar "energy" function such that this "energy" diminishes along the system's trajectory, then the system is said to be asymptotically stable. Since there is no need to solve the solution of the differential equations governing the system in determining its stability, it is usually referred to as the direct method (see *Lyapunov Stability*).

Although Lyapunov's direct method is efficient for stability analysis, it is of restricted applicability due to the difficulty in selecting a Lyapunov function. The situation is different when facing the controller design problem, where the control has not been specified, and the system under consideration is undetermined. Lyapunov functions have been effectively utilized in the synthesis of control systems. The basic idea is that, by first choosing a Lyapunov function candidate, a feedback control law can be specified such that it renders the derivative of the specified Lyapunov function candidate negative definite, or negative semi-definite when invariance principle can be used to prove asymptotic stability.

This way of designing control is called Lyapunov design. Lyapunov design depends on the selection of Lyapunov function candidate. Though the result is sufficient, it is a difficult problem to find a Lyapunov function (LF) satisfying the requirements of Lyapunov design. Fortunately, during the past several decades, many effective control design approaches have been developed for different classes of linear and nonlinear systems based on the basic ideas of Lyapunov design. Lyapunov functions are additive, like energy, i.e., Lyapunov functions for combinations of subsystems may be derived by adding the Lyapunov functions of the subsystems. This point can be seen clearly in the adaptive control design and backstepping design in this chapter.

Though Lyapunov design is a very powerful tool for control system design, stability and performance analysis, the construction of a Lyapunov function is not easy for general nonlinear systems, and it is usually a trial-and-error process owing to the lack of systematic methods. Different choices of Lyapunov functions may result in different control structures and control performance.

Past experience shows that a good design of Lyapunov function should fully utilize the property of the studied systems. Lyapunov design is used in many contexts, such as dynamic feedback, output-feedback, estimation of region of attraction, and adaptive control, among others. The chapter is not meant to be comprehensive but to serve as an introduction to the state-of-the-art of full-state feedback design based on Lyapunov techniques for several typical classes of autonomous systems.

Intensive research in adaptive control was first motivated by the design of autopilots for high performance aircraft in the early 1950s. Because their dynamics change drastically when they fly from one operating point to another, constant gain feedback control cannot handle it effectively. The lack of stability theory and one disastrous flight test led to the diminished interest in adaptive control in the late 1950s. The 1960s saw many advances in control theory and adaptive control in particular. Simultaneous development in computers and electronics made the implementation of complex controllers possible, and interest in adaptive control and its applications was renewed in the 1970s with several breakthrough results made.

The studies of non-robust behavior of adaptive control subject to small disturbance and unmodeled dynamics in 1979 and the early 1980s, led to better understanding of the instability mechanisms and the design of robust adaptive control in the later 1980s though it was very controversial initially. They were all systems satisfying the *matching condition*. In the late 1980s and early 1990s, the matching condition was relaxed to the *extended matching condition,* which for one period was regarded as the frontier that could not be crossed by Lyapunov design, and then further relaxed to the *strict-feedback systems* with general unmatched uncertainties through backstepping design, which is the state-of-the-art of adaptive control.

This chapter gives an overview of the state-of-the-art approaches of Lyapunov design and ways of choosing Lyapunov functions in the area of full-state adaptive control. Section 2 presents the concepts of Lyapunov stability analysis and control Lyapunov functions. In Section 3, Lyapunov functions for linear time invariant systems are presented first, then the results are utilized to solve Model Reference Adaptive Control (MRAC) problems for classes of linear and nonlinear systems which can be transformed to systems having stable linear portion.

For this class of problems, the choice of Lyapunov functions is systematic and controller design is standard. In Section 4, after the presentation of Lyapunov Redesign, Adaptive Lyapunov Redesign, Robust Lyapunov Redesign for a class of matched systems, backstepping controller design is discussed for unmatched nonlinear systems. By exploiting the physical properties of the systems under study, Section 5 shows that different choices of Lyapunov functions and better controllers are possible. Section 6 discusses the design flexibilities and considerations in actual applications of Lyapunov design, and further research work.

2. Control Lyapunov Function

Though Lyapunov's method applies to nonautonomous systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, for clarity and simplicity, we shall restrain our discussion to time-invariant nonlinear systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad (1)$$

where $\mathbf{x} \in R^n$, and $\mathbf{f}(\mathbf{x}) : R^n \to R^n$ is continuous. The basic idea of Lyapunov direct method consists of (i) choosing a radially unbounded positive definite Lyapunov function candidate $V(\mathbf{x})$, and (ii) evaluating its derivative $\dot{V}(\mathbf{x})$ along system dynamics (1) and checking its negativeness for stability analysis.

Lyapunov design refers to the synthesis of control laws for some desired closed-loop stability properties using Lyapunov functions for nonlinear control systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u}), \tag{2}$$

where $\mathbf{x} \in R^n$ is the state, $\mathbf{u} \in R^m$ is the control input, and $\mathbf{f}(\mathbf{x},\mathbf{u})$ is locally Lipschitz on $(\mathbf{x},\mathbf{u})$, and $\mathbf{f}(0,0) = 0$.

The usefulness of Lyapunov direct method for feedback control design $\mathbf{u}(\mathbf{x})$ can be seen as follows: Substituting $\mathbf{u} = \mathbf{u}(\mathbf{x})$ into (2), we have the autonomous closed-loop dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u}(\mathbf{x}))$ and Lyapunov direct method can then be used for stability analysis.

In actual applications, Lyapunov design can be conceptually divided into two steps:

(a) choose a candidate Lyapunov function V for the system, and
(b) design a controller which renders its derivative $\dot{V}$ negative.

Sometimes, it may be more advantageous to reverse the order of operation, i.e., design a controller that is most likely to be able to stabilize the closed-loop system first by examining the properties of the system, and then choose a Lyapunov function candidate V for the closed-loop system to show that it is indeed a Lyapunov function. Lyapunov design is sufficient. Stabilizing controllers are obtained if the processes succeed. If the attempts fail, no conclusion can be drawn on the existence of a stabilizing controller.

Let function $V(\mathbf{x})$ be a Lyapunov function candidate. Thus, the task is to search for $\mathbf{u}(\mathbf{x})$ to guarantee that, for all $\mathbf{x} \in R^n$, the time derivative of $V(\mathbf{x})$ along system (2) satisfy

$$\dot{V}(\mathbf{x}) = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x},\mathbf{u}(\mathbf{x})) \le -W(\mathbf{x}), \tag{3}$$

where $W(\mathbf{x})$ is a positive definite function. For affine nonlinear systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}, \ \ \mathbf{f}(\mathbf{0}) = \mathbf{0} \tag{4}$$

the inequality (3) becomes

$$\frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + \frac{\partial V}{\partial \mathbf{x}} \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) \le -W(\mathbf{x}). \tag{5}$$

In general, this is a difficult task. A system for which a good choice of $V(\mathbf{x})$ and $W(\mathbf{x})$

exists is said to possess a control Lyapunov function. A smooth positive definite and radially unbounded function $V(\mathbf{x}): R^n \to R_+$ is called a Control Lyapunov Function (CLF) for (2) if

$$\inf_{\mathbf{u}\in R^m}\{\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{u})\} < 0, \forall \mathbf{x} \neq \mathbf{0}. \quad (6)$$

If $V(\mathbf{x})$ is a CLF for affine nonlinear system (4), then a particular stabilizing control law, $\mathbf{u}(\mathbf{x})$, smooth for all $\mathbf{x} \neq \mathbf{0}$, is given by the Artstein and Sontag's universal controller

$$\mathbf{u}(\mathbf{x}) = \begin{cases} -\dfrac{\frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) + \sqrt{(\frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}))^2 + (\frac{\partial V}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x}))^4}}{\frac{\partial V}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})}, & \dfrac{\partial V}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x}) \neq 0 \\ 0, & \dfrac{\partial V}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x}) = 0 \end{cases} \quad (7)$$

The steps of Lyapunov design and concept of "Control Lyapunov Function" are used for systems with controls to differentiate the classical term "Lyapunov function" for systems without controls. As a design tool for general nonlinear systems, the main deficiency of the CLF concept is that a CLF is unknown. The task of finding an appropriate CLF may be as complex as that of designing a stabilizing feedback law. However, for several important classes of nonlinear systems, these two tasks can be solved simultaneously.

When $\dot{V}(\mathbf{x})$ is only negative semidefinite, asymptotic stability cannot be concluded from Lyapunov function method directly. However, if $\mathbf{x} = \mathbf{0}$ is shown to be the only solution for $\dot{V}(\mathbf{x}) = 0$, then asymptotic stability can still be drawn by evoking LaSalle's Invariance Principle, Invariant Set Theorem, which basically states that, if $\dot{V}(\mathbf{x}) \le 0$ of a chosen Lyapunov function candidate $V(\mathbf{x})$, then all solutions asymptotically converge to the largest invariant set in the set $\{\mathbf{x} \mid \dot{V}(\mathbf{x}) = 0\}$ as $t \to \infty$. In fact, this approach has been frequently used in the proof of asymptotic stability of a closed-loop system.

Lemma 2.1 [Barbalat] *Consider the function* $\phi(t): R_+ \to R$. *If* $\phi(t)$ *is uniformly continuous and* $\lim_{t\to\infty}\int_0^t \phi(\tau)d\tau$ *exists and is finite, then* $\lim_{t\to\infty}\phi(t) = 0$.

Theorem 2.1 [***LaSalle***] *Let* (a) Ω *be a positively invariant set of* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, ($b$) $V(\mathbf{x}): \Omega \to R_+$ *be a continuously differentiable function such that* $\dot{V}(\mathbf{x}) \le 0, \forall \mathbf{x} \in \Omega$, *and* ($c$) $E = \{\mathbf{x} \in \Omega \mid \dot{V}(\mathbf{x}) = 0\}$, *and* M *be the largest invariant set contained in* E. *Then, every bounded solution* $\mathbf{x}(t)$ *starting in* Ω *converges to* M *as* $t \to \infty$.

To show that a variable indeed converges to zero, Barbalat's Lemma is frequently used.

If Ω is the whole space R^n, then the above local Invariant Set Theorem becomes the global one. To prove the asymptotic stability of the system, we only need to show that no solution other than $\mathbf{x}(t) \equiv 0$ can stay forever in E.

It should be noted that there may exist many Lyapunov functions for a given nonlinear system. Specific choices of Lyapunov functions may yield better, cleaner controllers than others. Usually, Lyapunov functions are chosen as quadratic form due to its elegancy of mathematical treatment. However, it is not exclusive. Other forms have also been used in the literature, such as energy-based Lyapunov functions, integral-type Lyapunov functions, which have been applied in the design of controllers for classes of uncertain nonlinear systems.

3. Lyapunov Design via Lyapunov Equation

Model Reference Adaptive Control (MRAC) was originally proposed to solve the problem in which the design specifications are given by a reference model, and the parameters of the controller are adjusted by an adaptation mechanism/law such that the closed-loop dynamics of the system are the same as the reference model which gives the desired response to a command signal. In solving this class of problems, the Lyapunov equation plays a very important role in choosing the Lyapunov function and deriving the feedback control and adaptation mechanism.

In fact, the construction of Lyapunov functions is systematic and straightforward for the class of systems which can be transformed into systems with two portions: (i) a stable linear portion so that linear stability results can be directly applied, and (ii) matched nonlinear portion which can be handled using different techniques such as adaptive or robust control techniques in different situations.

Thus, MRAC can also be viewed as Lyapunov design based on Lyapunov equations. To explain the concepts clearly, Lyapunov equation and Lyapunov stability analysis are firstly presented for linear time-invariant systems, then adaptive control design for classes of unknown linear time invariant systems and unknown nonlinear systems is presented by utilizing Lyapunov equation.

3.1. Lyapunov Equation

Though linear systems are well understood, it is interesting to look at them in the Lyapunov language, and provide a basis of Lyapunov design for systems having linear portions. For simplicity, consider the following simple controllable Linear Time Invariant (LTI) systems described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \,, \tag{8}$$

where $\mathbf{x} \in R^n$, and $u \in R$ are the states, and control variable, respectively, $\mathbf{A} \in R^{n \times n}$ and $\mathbf{b} \in R^n$. It is well known that there is always a global quadratic LF, and the stabilizing controller can be obtained constructively. Let the state feedback control be

$$u = -\mathbf{kx} \tag{9}$$

the resulting closed-loop system will be of the form

$$\dot{\mathbf{x}} = \mathbf{A}_m\mathbf{x},\ \ \mathbf{A}_m = \mathbf{A} - \mathbf{bk}\ . \tag{10}$$

From the linear system theory, there are many ways to design $\mathbf{k}$ for a desirable stable closed-loop system. The most intuitive and direct one might be the pole-placement method. In the context of this chapter, we shall look at the problem in the sense of Lyapunov design. Not surprisingly, Lyapunov functions can be systematically found to describe stable linear systems owing to the following theorem.

Theorem 3.1 *The LTI system* $\dot{\mathbf{x}} = \mathbf{A}_m\mathbf{x}$ *is asymptotically stable if and only if, given any symmetric positive-definite matrix* $\mathbf{Q}$*, there exists a symmetric positive-definite matrix* $\mathbf{P}$*, which is the unique solution of the so-called Lyapunov equation*

$$\mathbf{PA}_m + \mathbf{A}_m^T\mathbf{P} = -\mathbf{Q}\ . \tag{11}$$

For such a solution, the positive definite quadratic function of the form

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{Px} \tag{12}$$

is a LF for the closed-loop system (10), since

$$\dot{V}(\mathbf{x}) = -\mathbf{x}^T\mathbf{Qx} < 0,\ \ \forall\mathbf{x} \neq 0\ . \tag{13}$$

Another method to design $\mathbf{k}$ is the well known optimal linear quadratic (LQ) design method. To investigate the problem in the context of CLF, consider the following Lyapunov function candidate for (8)

$$V = \mathbf{x}^T\mathbf{Px}\ , \tag{14}$$

where $\mathbf{P} = \mathbf{P}^T > 0$. For $\mathbf{P}$ to define a CLF (6), the following inequality should hold

$$\inf_{u\in R}\{\mathbf{x}^T\mathbf{A}^T\mathbf{Px} + \mathbf{x}^T\mathbf{PAx} + u^T\mathbf{b}^T\mathbf{Px} + \mathbf{x}^T\mathbf{Pb}u\} < 0\ \ \forall\mathbf{x} \neq \mathbf{0}\ , \tag{15}$$

which implies that $u(\mathbf{x})$ should take the form that $u(\mathbf{x}) = -\gamma\mathbf{b}^T\mathbf{Px}$ with $\gamma > 0$ (the corresponding linear feedback gain $\mathbf{k} = \gamma\mathbf{b}^T\mathbf{P}$). Thus, $\mathbf{P}$ defines a CLF if

$$\inf_{\gamma\in R}\{\mathbf{A}^T\mathbf{P} + \mathbf{PA} - 2\gamma\mathbf{Pbb}^T\mathbf{P}\} < 0\ . \tag{16}$$

Such a $\mathbf{P}$ can always be found through the solution of algebraic Riccati equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} + \mathbf{Q} - 2\gamma\mathbf{Pbb}^T\mathbf{P} = 0\,, \tag{17}$$

which is known to solve the optimal linear quadratic (LQ) state design which minimizes the cost function $J = \int_0^\infty [\mathbf{x}^T\mathbf{Qx} + \frac{1}{2\gamma}u^2]dt$ and subject to the dynamic constraints imposed by (8) (see *Optimal Linear Quadratic Control* (*LQ*)). Equation (17) guarantees that $\dot{V} = -\mathbf{x}^T\mathbf{Qx}$, and in turn asymptotic stability of the closed-loop systems.

The results are readily available for multi-input-multi-output (MIMO) systems. Techniques in dealing with linear systems in state space are well established. (see *Classical Design Methods for Continuous LTI-Systems*, *Design of State Space Controllers* (*Pole Placement*) *for SISO Systems*, *Pole Placement Control*, *Optimal Linear Quadratic Control* (*LQ*)).

3.2. MRAC for Linear Time Invariant Systems

To illustrate the basic steps in solving MRAC for linear time invariant systems, consider the following LTI plant described by the state-space model

$$\dot{\mathbf{x}} = \mathbf{Ax} + g\mathbf{b}u\,, \tag{18}$$

where $\mathbf{x} \in R^n$; $u \in R$ are the states and input respectively, $\mathbf{A} \in R^{n\times n}$ and $\mathbf{b} \in R^n$ are in the controller canonical form as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & \cdots & -a_n \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{19}$$

with unknown constants $a_i, i = 1,\cdots,n$, and control input gain $g > 0$ is an unknown constant. The objective is to drive $\mathbf{x}$ to follow some desired reference trajectory $\mathbf{x}_m \in R^n$ and guarantee closed-loop stability. Let the reference trajectory $\mathbf{x}_m$ be generated from a reference model specified by the LTI system

$$\dot{\mathbf{x}}_m = \mathbf{A}_m\mathbf{x}_m + g_m\mathbf{b}r\,, \tag{20}$$

where $r \in R$ is a bounded reference input, $\mathbf{A}_m \in R^{n\times n}$ is a stable matrix given by

$$\mathbf{A}_m = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_{m1} & -a_{m2} & \cdots & -a_{mn} \end{bmatrix} \tag{21}$$

with $a_{mi}, i = 1, \cdots, n$, chosen such that $s^n + a_{mn}s^{n-1} + \cdots + a_{m1}$ is a Hurwitz polynomial. The reference model and input r are chosen such that $\mathbf{x}_m(t)$ represents a desired trajectory that $\mathbf{x}$ has to follow, i.e., $\mathbf{x} \to \mathbf{x}_m$ as $t \to \infty$.

Consider a general linear control law of the form

$$u = \mathbf{k}(t)\mathbf{x} + k_r(t)r \,, \tag{22}$$

where $\mathbf{k}$ and k_r may be chosen freely. The closed-loop system then becomes

$$\dot{\mathbf{x}} = (\mathbf{A} + g\mathbf{bk})\mathbf{x} + gk_r\mathbf{b}r \,. \tag{23}$$

It is clear that there exist constant parameters $\mathbf{k}^*$ and k_r^* such that the *matching conditions*

$$a_i + gk_i^* = a_{mi}, \quad gk_r^* = g_m \tag{24}$$

hold, i.e., equations (20) and (23) are equivalent. Since a_i and g are unknown, so are $\mathbf{k}^*$ and k_r^*, which means that controller (22) with $\mathbf{k} = \mathbf{k}^*$ and $k_r = k_r^*$ is not feasible. This problem can be easily solved using on-line adaptive control techniques.

For ease of discussion, let $\boldsymbol{\theta} = [\mathbf{k}^*, k_r^*]^T$, $\hat{\boldsymbol{\theta}} = [\mathbf{k}(t), k_r(t)]^T$, define parameter estimation errors $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}} = [\tilde{\boldsymbol{\theta}}_x^T, \tilde{\boldsymbol{\theta}}_r]^T$ with $\tilde{\boldsymbol{\theta}}_x = \mathbf{k}^* - \mathbf{k}(t)$, $\tilde{\theta}_r = k_r^* - k_r(t)$ and denote $\phi = [\mathbf{x}^T, r]^T$. Accordingly, equation (23) can be written as

$$\dot{\mathbf{x}} = (\mathbf{A} + g\mathbf{bk}^*)\mathbf{x} + g\mathbf{b}k_r^*r - g\mathbf{b}\tilde{\boldsymbol{\theta}}_x\mathbf{x} - g\mathbf{b}\tilde{\theta}_r r$$

$$= \mathbf{A}_m\mathbf{x} + g_m\mathbf{b}r - g\mathbf{b}\phi^T\tilde{\boldsymbol{\theta}} \,. \tag{25}$$

Define the tracking error $\mathbf{e} = \mathbf{x} - \mathbf{x}_m$. Comparing equations (20) and (25) give the closed-loop error equation

$$\dot{\mathbf{e}} = \mathbf{A}_m\mathbf{e} - g\mathbf{b}\phi^T\tilde{\boldsymbol{\theta}} \,, \tag{26}$$

which has a stable linear portion and an unknown parametric uncertainty input, which turns out to be easily solvable using the facts that (i) given any stable known matrix $\mathbf{A}_m$, for any symmetric positive-definite matrix $\mathbf{Q}$, there exists a unique symmetric positive-definite matrix $\mathbf{P}$ satisfying

$$\mathbf{PA}_m + \mathbf{A}_m^T\mathbf{P} = -\mathbf{Q} \tag{27}$$

as detailed in Subsection 3.1, and (ii) the linear-in-the-parameter uncertainty $\mathbf{b}\phi^T\tilde{\boldsymbol{\theta}}$ can be dealt with using adaptive techniques. Owing to the above observations, choose the Lyapunov function candidate by augmenting the Lyapunov function in (11) with a quadratic parameter estimation error term as follows

$$V(\mathbf{e},\boldsymbol{\theta}) = \mathbf{e}^T\mathbf{P}\mathbf{e} + g\tilde{\boldsymbol{\theta}}^T\Gamma^{-1}\tilde{\boldsymbol{\theta}},\ \Gamma = \Gamma^T > \mathbf{0}. \tag{28}$$

Noticing that $\dot{\tilde{\boldsymbol{\theta}}} = -\dot{\hat{\boldsymbol{\theta}}}$, the time derivative of V is given by

$$\dot{V} = -\mathbf{e}^T\mathbf{Q}\mathbf{e} - 2g\mathbf{e}^T\mathbf{P}\mathbf{b}\phi^T\tilde{\boldsymbol{\theta}} + 2g\tilde{\boldsymbol{\theta}}^T\Gamma^{-1}\dot{\tilde{\boldsymbol{\theta}}}$$

$$= -\mathbf{e}^T\mathbf{Q}\mathbf{e} - 2g\tilde{\boldsymbol{\theta}}^T\Gamma^{-1}(\Gamma\phi\mathbf{e}^T\mathbf{P}\mathbf{b} - \dot{\hat{\boldsymbol{\theta}}}). \tag{29}$$

Apparently, choosing the parameter adaptation law as

$$\dot{\hat{\boldsymbol{\theta}}} = -\Gamma\phi\mathbf{e}^T\mathbf{P}\mathbf{b} \tag{30}$$

leads to $\dot{V} = -\mathbf{e}^T\mathbf{Q}\mathbf{e} \le 0$. Accordingly, the following conclusions are in order: (i) the boundedness of $\mathbf{e}$ and $\tilde{\boldsymbol{\theta}}$, (ii) the boundedness of $\mathbf{x}$ and $\hat{\boldsymbol{\theta}}(t)$ (i.e., $\mathbf{k}(t)$ and $k_r(t)$) by noting the boundedness of $\mathbf{x}_m$ and $\boldsymbol{\theta}$, and the boundedness of the control signal u, and (iii) the tracking error $\lim_{t\to\infty}\mathbf{e}\to\mathbf{0}$ using Barbalat Lemma 1 because (a) $\int_0^\infty \mathbf{e}^T\mathbf{e} < c$ with constant $c > 0$ obtainable from (30), and (b) $\mathbf{e}$ is uniformly continuous since $\dot{\mathbf{e}}$ is bounded as can be seen from equation (26).

The basic ideas are not only readily applicable to MIMO LTI systems, they can also be extended for a class of nonlinear systems as will be detailed next.

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